GEOMETRIC GROUP THEORY

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1. Preliminaries

1.1. Topology. We firstly recall basics from topology.

Definition 1.1. A *topological space* is a pair $\langle X, \mathscr{T} \rangle$, where X is a set and $\mathscr{T}$ is a collection of subsets of X (called a *topology* on X), satisfying the following properties:

(1) if $\mathscr{U} \subseteq \mathscr{T}$, then $\bigcup \mathscr{U} \in \mathscr{T}$;
(2) if $\mathscr{V} \subseteq \mathscr{T}$ is finite, then $\bigcap \mathscr{V} \in \mathscr{T}$.

The elements of $\mathscr{T}$ are called *open sets*, while their complements are called *closed sets*.

Note. If X is a topological space, $\varnothing$ is both closed and open (clopen).

Example 1.2. For each set X, one may define the *discrete topology*, for which every subset of X is open. On the other hand, for each set X, one may define the *indiscrete topology*, in which only $\varnothing$ and X are open.

Definition 1.3. Let Y be a subset of a topological space X. The *subspace topology* on Y, is the topology for which $V \subseteq Y$ is open in Y if, and only if, $V = U \cap Y$ for some open $U \subseteq X$ in X.

Definition 1.4. Let X be a topological space, and $\sim$ be an equivalence relation on X. Let $Y = X/_{\sim}$. We call $q : X \to Y$ be defined by $x \mapsto [x]$ for each $x \in X$ the *quotient map*. We call Y a *quotient space*, where U is open in Y if, and only if, $q^{-1}(U)$ is open in X.

Example 1.5. Consider $X = \mathbb{R}$ with the usual Euclidean topology, and the equivalence relation $\sim$ where $x \sim y$ if, and only if, $x - y \in \mathbb{Z}$. Then, $X/_{\sim} = \mathbb{R}/\mathbb{Z}$, which yields a circle S^1, as depicted in Figure 1.

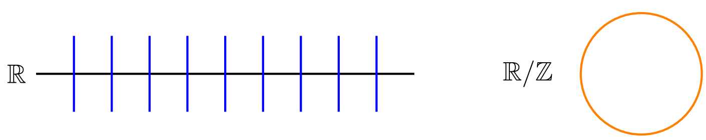


Figure 1. Quotient space in Example 1.5

Definition 1.6. A *path* in a topological space X, is a continuous map $p : [0,1] \to X$. If $p(0) = x$ and $p(1) = y$, we say p is a *path from x to y*, as depicted in Figure 2. If there is a path between every two points in our space, we say the topological space is *path-connected*. A *loop based at x_0*, is a continuous map $\alpha : [0,1] \to X$ such that $\alpha(0) = \alpha(1) = x_0$, as depicted in Figure 3.

Definition 1.7. A *pointed space* is a pair (X, x_0), where X is a topological space and $x_0 \in X$ is a distinguished point of X, called the *base point*.

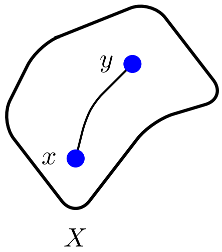


FIGURE 2. A path between x and y in X

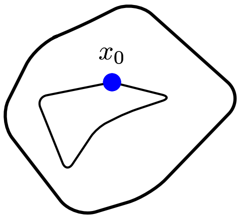


FIGURE 3. A loop of x_0 in X

Definition 1.8. Two maps $f, g : X \to Y$ are *homotopic* to each other, if there is a continuous function $F : X \times [0, 1] \to Y$, for which $F(x, 0) = f(x)$ and $F(x, 1) = g(x)$ for each $x \in X$. Informally, f and g are homotopic if f can be continuously deformed into the function g. We will write $f \sim g$ to denote that f and g are homotopic, which defines an equivalence relation.

Example 1.9. We will consider *homotopy of paths*, as depicted in Figure 4. That is to say, given two paths $p, p' : [0, 1] \to X$ from x to y, we consider a continuous map $F : I \times I \to X$ with $F(t, 0) = p(t)$ and $F(t, 1) = p'(t)$.

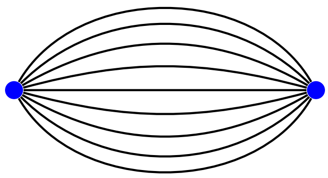


FIGURE 4. Homotopy of paths

Let (X, x_0) be a pointed space. Given two loops based at x_0, say α and β, we can form a new loop at x_0, denoted $\alpha \cdot \beta$ (called *loop composition*) by first following α (at twice speed), then following β at twice the speed. More precisely:

$$(\alpha \cdot \beta)(t) = \begin{cases} \alpha(2s) & \text{if } 0 \leq s \leq \frac{1}{2}; \\ \beta(2s - 1) & \text{if } \frac{1}{2} \leq s \leq 1; \end{cases}$$

for each $t \in [0, 1]$. Define a relation $\sim$ on loops based at x_0 as follows. We say that γ is *homotopic to γ' relative to* $0, 1$, denoted $\gamma \sim \gamma'$ rel $(0, 1)$, if there is a homotopy $F : I \times I \to X$ with: $F(s, 0) = \gamma(s)$, $F(s, 1) = \gamma'(s)$ and $F(0, t) = F(1, t) = x_0$ for all $s, t \in [0,1]$. In other words, we can continuously deform the loop γ into the loop γ' while keeping the start and endpoints of the loop fixed at x_0.

The relation $\sim$ rel $(0,1)$ defines an equivalence relation, so we let $[\gamma]$ denote the equivalence class of γ. Loop composition induces a well-defined and associative binary operation on the set of equivalence classes of loops based at x_0, by defining $[\alpha] \cdot [\beta] = [\alpha \cdot \beta]$. Moreover, there is an identity element $[\iota]$ for this binary operation (take the constant map to x_0), and each element $[\gamma]$ has an inverse. Thus, the equivalence classes of loops based at x_0 forms a group under this binary operation. This group is denoted by $\pi_1(X, x_0)$, and is called the *fundamental group* of (X, x_0). Indeed, the fundamental group need not be Abelian.

Note. If X is path-connected and $x_0, x_1 \in X$, then $\pi_1(X, x_0) \cong \pi_1(X, x_1)$. Hence, if X is path-connected, we write $\pi_1(X)$ for the fundamental group of X.

Example 1.10. The fundamental group of the circle is the integers, i.e., $\pi_1(S^1) = \mathbb{Z}$. Winding around the circle twice is different to winding around the circle only once, due to difference of angle (4π in former case, 2π in latter case).

Definition 1.11. A topological space is *simply connected*, if X is path-connected and $\pi_1(X)$ is the trivial group.

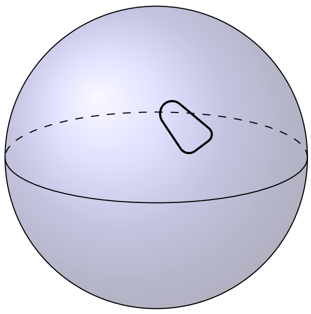


FIGURE 5. S^2 is simply connected

There are many applications of Algebraic Topology, including the Fundamental Theorem of Algebra, and Brouwer's Fixed Point Theorem.

1.2. Topological Groups.

Definition 1.12. A group equipped with a topology such that multiplication and inversion are continuous maps is called a *topological group*.

Example 1.13. There are many examples of topological groups, including $(\mathbb{R}^n, +)$ and $(\mathbb{C}^n, +)$. Embedding S^1 into the complex plane, we obtain that is forms a topological group under complex multiplication.

Example 1.14. The *general linear group*, denoted $\mathrm{GL}(n, \mathbb{C})$, is the group of $n \times n$ invertible matrices with complex entries. The *orthogonal group*, denoted $O(n)$, is the group $\{M \in \mathrm{GL}(n, \mathbb{C}) \mid M^{-1} = M^T\}$. The *special linear group*, denoted $\mathrm{SL}(n, \mathbb{C})$, is the group $\{M \in \mathrm{GL}(n, \mathbb{C}) \mid \det M = 1\}$. The *unitary group*, denoted $U(n)$, is the group $\{M \in \mathrm{GL}(n, \mathbb{C}) \mid U^{-1} = U^*\}$, where $*$ is the conjugate transpose. The *special orthogonal group*, denoted $SO(n)$, is the group $O(n) \cap \mathrm{SL}(n, \mathbb{C})$.

Note. We equip $\mathrm{GL}(n, \mathbb{C})$ with Euclidean metric topology, embedded into $\mathbb{C}^{n^2}$. Subgroups are equipped with the subspace topology. Then norm of complex $n \times n$

matrix $A = (a_{ij})$ is defined to be the real number $|A| = \left(\sum_{i,j=1}^{n} |a_{ij}|^2\right)^{\frac{1}{2}}$. The metric is therefore $d(A, B) = |A - B|$.

Theorem 1.15. $\mathrm{GL}\,(n, \mathbb{C})$ *is a topological group.*

Proof. Matrix multiplication $(A, B) \mapsto AB$ is continuous, since entries of AB are polynomial in the entries of A and B. The determinant function

$$\det : \mathrm{GL}\,(n, \mathbb{C}) \to \mathbb{C}^*$$

is continuous, since $\det A$ is a polynomial in the entries of A. By the adjoint formula for A^{-1}, we have

$$\left(A^{-1}\right)_{ji} = (-1)^{i+j} \frac{\det A^{ij}}{\det A},$$

where A^{ij} is the matrix obtained from A by deleting the ith row and the jth column. Consequently, each entry of A^{-1} is a rational function of the entries of A. Therefore, the inversion map $A \mapsto A^{-1}$ is continuous. Thus, $\mathrm{GL}\,(n, \mathbb{C})$ is a topological group. □

Theorem 1.16. *The unitary group $U(n)$ is compact and connected.*

Proof. We claim $U(n)$ is closed and bounded. To see why, firstly observe that if $A \in U(n)$, then $|A|^2 = \sum_{j=1}^{n} |Ae_j|^2 = n$; for A is unitary, and thus an isometry with respect to usual norm (so $|Ax| = |x|$ for each $x \in \mathbb{C}^n$). Therefore, $U(n)$ is a bounded subset of $\mathbb{C}^{n^2}$. To see why $U(n)$ is closed, we observe the self-map $f : \mathbb{C}^{n^2} \to \mathbb{C}^{n^2}$ defined by $f(A) = A^*A$ is continuous (where $*$ is conjugate transpose). Therefore, $U(n) = f^{-1}(I)$ is a closed subset of $\mathbb{C}^{n^2}$. Hence, $U(n)$ is a closed and bounded subset of $\mathbb{C}^{n^2}$, and consequently compact.

Now, we show $U(n)$ is connected. If $A \in U(n)$, then A can be diagonalised by another unitary matrix S. Any diagonal unitary matrix must have complex numbers of absolute value 1 on the main diagonal (recall it is isometry with respect to usual norm). We can therefore write $A = S\mathrm{diag}\big(e^{i\theta_1}, \dots, e^{i\theta_n}\big)S^{-1}$. A path in $U(n)$ from I to A is then given by $t \mapsto S\mathrm{diag}\big(e^{ti\theta_1}, \dots, e^{ti\theta_n}\big)S^{-1}$. □

Note. The unitary group is not, however, simply connected; for its fundamental group is the $\mathbb{Z}$ for every n. That is, $\pi_1(U(n)) \cong \mathbb{Z}$.

Corollary 1.17. *The orthogonal group $O(n)$ is compact.*

Proof. Since $\mathbb{R}^{n^2}$ is closed in $\mathbb{C}^{n^2}$ and $O(n) = U(n) \cap \mathbb{R}^{n^2}$, we have that $O(n)$ is closed in $U(n)$, and so $O(n)$ is compact. □

Definition 1.18. Given a group G acting on a space X, written $G \curvearrowright X$, we consider the *orbit space*, where we define an equivalence relation $\sim$ on X by being in the same orbit. That is to say, given $x, y \in X$, $x \sim y$ if, and only if, $y \in \{gx \mid g \in G\}$ and $x \in \{gy \mid g \in G\}$.

Note. Typically, we consider action by isometries (such as on $\mathbb{R}$ via translation).

Definition 1.19. A *discrete group*, is a topological group in which all points are isolated (i.e., it is equipped with the discrete topology).

Proposition 1.20. *A topological group G is a discrete group if, and only if, the identity element of G is isolated.*

Proof. The forward direction is trivial. Conversely, suppose $1 \in G$ is isolated. Fix $g \in G$. Consider the map defined by $h \mapsto hg^{-1}$ for each $h \in G$. Such a map is continuous, since multiplication is continuous. In fact, it defines a homeomorphism. Observe the pre-image of 1 under this map is g, which implies g is isolated. Thus, each point in G are isolated, and we are done. □

Example 1.21. Interesting examples include discrete subgroups of Lie groups, or matrix groups. For example, $\mathrm{SL}(n,\mathbb{Z}) \subseteq \mathrm{SL}(n,\mathbb{R})$, $\mathbb{Z} \subseteq \mathbb{R}$ and $\mathbb{Z}[i] \subseteq \mathbb{C}$, where the latter is the Guassian integers. The last two examples are depicted in Figure 6.

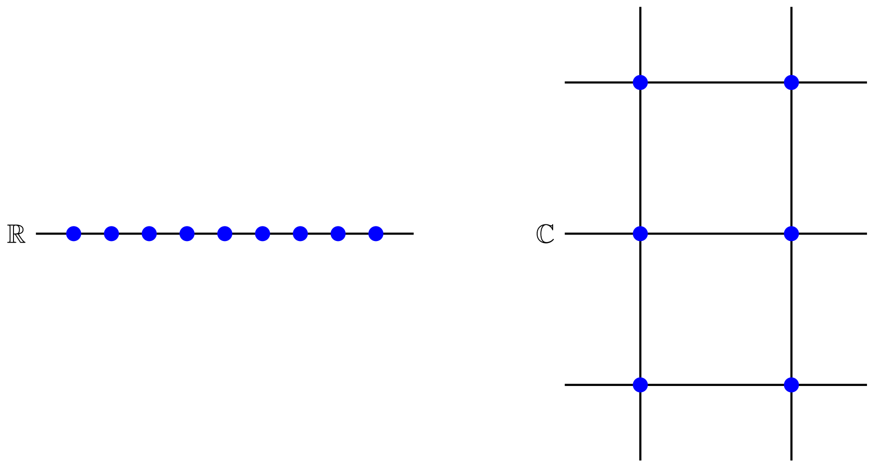


FIGURE 6. Discrete subgroups of $\mathbb{R}$ and $\mathbb{C}$ in Example 1.21

Theorem 1.22. *A subgroup Γ of $\mathbb{R}^n$ is discrete if, and only if, Γ is generated by a set of linearly independent vectors. If $\Gamma \leq \mathbb{R}^n$ is discrete, then Γ is countable.*

Proof. Suppose $\Gamma \leq \mathbb{R}^n$ is discrete. If not, we can construct a decreasing (w.r.t. norm) sequence of distinct points, contradicting limit is isolated. Conversely, suppose Γ is generated by a set of linearly independent vectors, say $\{x_1, \ldots, x_m\}$ for some $x_1, \ldots x_m \in \mathbb{R}^n$. Then, $x \in \Gamma$ if, and only if, $x = \alpha_1 x_1 + \ldots + \alpha_m x_m$ where $\alpha_i \in \mathbb{Z}$ for each $i \in \{1, \ldots, m\}$. Therefore, $\|x\|^2 = \sum_{i=1}^m |\alpha_i| s_i$, where $s_i = \|x_i\|$. So, $\|x\|^2 \geq \min_{i\in[m]} s_i > 0$, which implies 1 is isolated and so Γ is discrete. □

Definition 1.23. A group G is *finitely generated*, if $G = \langle S \rangle$, where $S \subseteq G$ is finite. The inclusion map $i : S \hookrightarrow G$ extends uniqely to an epimorphism (surjection) $\pi_S : F(S) \to G$, where $F(S)$ is the free group generated by S. We have the kernel is "normally generated by R", $\ker(\pi_S) = \langle\langle R \rangle\rangle := \bigcap_{R \subseteq N \lhd G} N$. If R is finite, we say G is finitely presented. We write $G = \langle S \mid R \rangle$.

Note. A group which is not finitely generated is $(\mathbb{Q}, +)$.

Example 1.24. Consider integers modulo 6, i.e., $\mathbb{Z}/6\mathbb{Z} = \langle x, y \mid x^2, y^3, [x,y] \rangle$. Then $x = 3$ and $y = 2$. Consider $\mathbb{Z}^n = \langle a_1, \ldots, a_n \mid [a_i, a_j] = 1\ \forall 1 \leq i \neq j \leq n \rangle$.

Lemma 1.25 (Ping Pong Lemma)**.** *Suppose $G \curvearrowright X$, and that G_1 and G_2 are subgroups of G with $|G_1| \geq 2$ and $|G_2| \geq 3$. Let X_1 and X_2 be subsets of X such that $X_1 \not\subseteq X_2$ and $X_2 \not\subseteq X_1$. Suppose $(G_1 \setminus \{1\})X_2 \subseteq X_1$ and $(G_2 \setminus \{1\})X_1 \subseteq X_2$. Then, $\langle G_1 \cup G_2 \rangle \cong G_1 * G_2$.*

Proof. Let $\phi_1 : G_1 \hookrightarrow \langle G_1 \cup G_2 \rangle$ and $\phi_2 : G_2 \hookrightarrow \langle G_1 \cup G_2 \rangle$ be inclusion maps (which are homomorphisms). Then by the universal property of free products, there exists homomorphism $h : G_1 * G_2 \to \langle G_1 \cup G_2 \rangle$ such that $\phi|_{G_i} = G_i$ for each $i \in \{1, 2\}$. In particular, ϕ is onto. We now show the kernel of ϕ is trivial, which will finish the proof.

To this end, suppose $w \in \ker(\phi) \leq G_1 * G_2$. We consider the unique reduced form of w. There are four cases to consider.

Case 1: Suppose $w = a_1b_1\dots a_nb_na_{n+1}$ where $a_i \in G_1 \setminus \{1\}$ and $b_j \in G_2 \setminus \{1\}$. Suppose $x_2 \in X_2 \setminus X_1$. Then,

$$x_2 = \phi(w)\cdot x_2 = a_1\cdot(b_1\cdot\dots\cdot(a_n\cdot(b_n\cdot(a_{n+1}\cdot x_2)))) \in X_1,$$

since $a_{n+1}\cdot x_2 \in X_1$, $b_n\cdot(a_{n+1}\cdot x_2) \in X_2$, and so on, but this yields a contradiction.

Case 2: Suppose $w = b_1a_1b_2a_2\dots b_na_nb_{n+1}$ where $a_i \in G_1 \setminus \{1\}$ and $b_j \in G_2 \setminus \{1\}$. Suppose $x_1 \in X_1 \setminus X_2$. Then,

$$x_1 = \phi(w)\cdot x_1 = b_1\cdot(a_1\cdot\dots\cdot(b_n\cdot(a_n\cdot(b_{n+1}\cdot x_1)))) \in X_2,$$

since $b_{n+1}\cdot x_1 \in X_2$, $a_n\cdot(b_{n+1}\cdot x_1) \in X_1$, and so on, but this yields a contradiction.

Case 3: Suppose $w = a_1b_1\dots a_nb_n$ where $a_i \in G_1 \setminus \{1\}$ and $b_j \in G_2 \setminus \{1\}$. Since $|G_2| \geq 3$, there exists $b \in G_2 \setminus \{1, b_n\}$. Then

$$bwb^{-1} = ba_1b_1a_2b_2\dots a_n\overbrace{(b_nb^{-1})}^{\in G_2\setminus\{1\}}$$

is reduced and $bwb^{-1} \in \ker(\phi)$. However, we obtain a contradiction by case 2.

Case 4: Suppose $w = b_1a_1b_2a_2\dots b_na_n$ where $a_i \in G_1\setminus\{1\}$ and $b_j \in G_2\setminus\{1\}$. Then, $b_1wb_1^{-1}$ is a reduced word in the kernel of ϕ. But we obtain a contradiction by case 3.

Thus, it follows $\langle G_1 \cup G_2\rangle \cong G_1 * G_2$. □

Note. Suppose $G \curvearrowright X$, and we have $g, h \in G$ and $x \in X$. Say

- $g(X \setminus A^-) \subseteq A^+$;
- $g^{-1}(X \setminus A^+) \subseteq A^-$;
- $h(X \setminus B^-) \subseteq B^+$;
- $h^{-1}(X \setminus B^+) \subseteq B^-$.

Then consider Figure 7, demonstrating action of ghg^{-1}.

Example 1.26. Let $A = \begin{pmatrix}1 & 2\\ 0 & 1\end{pmatrix}$ and $B = \begin{pmatrix}1 & 0\\ 2 & 1\end{pmatrix}$. Consider words in A and B, such as $A^5B^{-7}A^9B^{13}$. We claim $w(A,B) \neq I$, that is, there are no non-trivial relations yielding the identity for words in A and B. In other words, A and B freely generate a subgroup of $\mathrm{SL}(2,\mathbb{Z})$. We prove this via the Ping Pong Lemma.

Let $G_1 = \langle A\rangle = \left\{\begin{pmatrix}1 & 2n\\ 0 & 1\end{pmatrix} \,\middle|\, n \in \mathbb{Z}\right\}$ and $G_2 = \langle B\rangle = \left\{\begin{pmatrix}1 & 0\\ 2n & 1\end{pmatrix} \,\middle|\, n \in \mathbb{Z}\right\}$. Note $\mathrm{SL}(2,\mathbb{Z}) \curvearrowright \mathbb{P}(\mathbb{R}^2)$ projective space. Let

$$X_1 = \left\{\begin{pmatrix}x\\ y\end{pmatrix} \,\middle|\, |y| \leq |x|\right\}$$

and

$$X_2 = \left\{\begin{pmatrix}x\\ y\end{pmatrix} \,\middle|\, |y| \geq |x|\right\},$$

which are depicted in Figure 8 (with X_1 straight line region and X_2 wavy line region).

Consider $\begin{pmatrix}1 & 2n\\ 0 & 1\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix} = \begin{pmatrix}x+2ny\\ y\end{pmatrix}$, where $|y| \geq |x|$. Then,

$$\begin{aligned}|x+2ny| &\geq |2n||y| - |x|\\ &\geq |y| + |y| - |x|\\ &\geq |y|.\end{aligned}$$

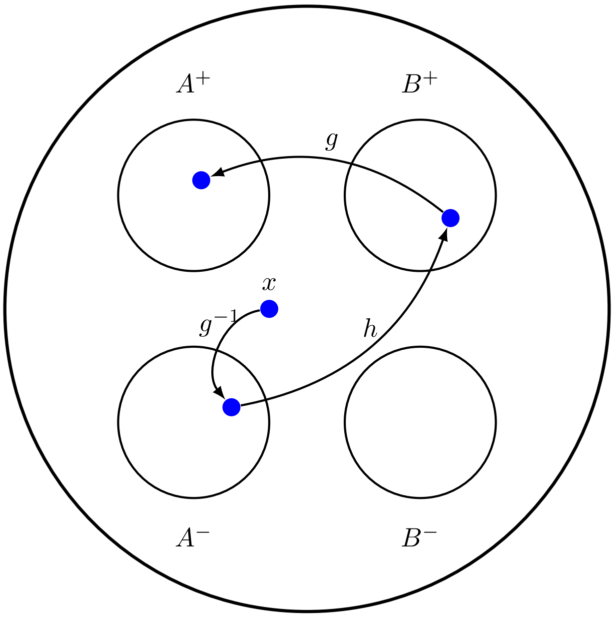


FIGURE 7. Verification of Ping Pong Lemma

Hence, $(G_1 \setminus \{1\})X_2 \subseteq X_1$.

Similarly, we consider $\begin{pmatrix} 1 & 0 \\ 2n & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ 2nx + y \end{pmatrix}$, where $|y| \leq |x|$. Then,

$$\begin{aligned} |2nx + y| &\geq |2n||x| - |y| \\ &\geq |x| + |x| - |y| \\ &\geq |x|. \end{aligned}$$

Hence, $(G_2 \setminus \{1\})X_1 \subseteq X_2$. Thus, by the Ping Pong Lemma,

$$\langle A, B \rangle \cong G_1 * G_2 \cong \mathbb{Z} * \mathbb{Z} \cong F_2.$$

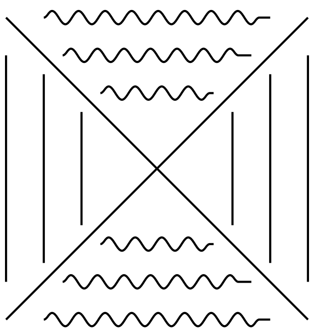


FIGURE 8. Ping Pong of words in A and B

2. GEOMETRY

2.1. **Tilings.** We now consider tilings of $\mathbb{R}^2$, as depicted on the left in Figure 9. We may define an equivalence relation $\sim$, where $x \sim y$ if, and only if, $x = y$ or

$x, y \in \partial R$ and $x = w(y)$ or $y \in w(x)$ for some $w \in \{f, g, fg, f^{-1}g\}$ (we don't need to list the rest, since $[f, g] = 1$; see Figure 12). For example, see right hand side of Figure 9.

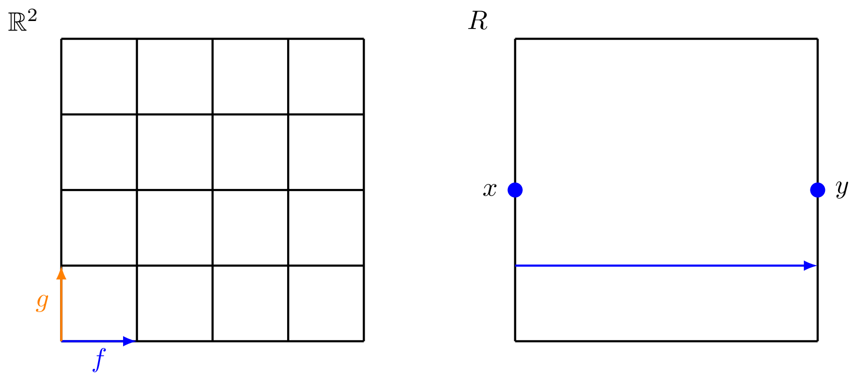


FIGURE 9. Tiling of $\mathbb{R}^2$

Question 2.1. What do we get if we identify two edges of our tiling?

We obtain a torus $S^1 \times S^1$ (see Figure 10).

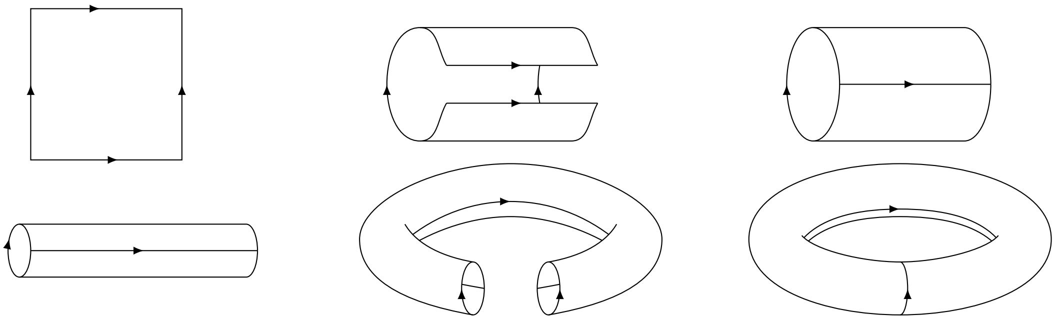


FIGURE 10. Torus construction

Note. Recall a *homeomorphism*, is a continuous open bijection $f : U \to V$. For example, take a look at Figure 11, where the spaces are actually homotopy equivalent.

FIGURE 11. Homotopic spaces

Proposition 2.2. $R/_\sim \cong \mathbb{R}^2/\langle f,g\rangle \cong \mathbb{T}^2$.

Proof. We may embed R into $\mathbb{R}^2$, since we tile $\mathbb{R}^2$ by R. Let $q : \mathbb{R}^2 \to R/_\sim$ be defined by $x \mapsto [w(x)]$, for each $x \in \mathbb{R}^2$, where $w \in \langle f,g\rangle$ such that $w(x) \in R$. Note for each $x \in \mathbb{R}^2$ such a w exists. For recall we embed R as one of the tiles, and x lies in some tile T; via translations (using words in f and g), we can map T to R (and in particular, x to $w(x) \in R$). Now that we have established the map is well-defined, we observe the mapping is onto (we are viewing R embedded in $\mathbb{R}^2$). Now, it is clear $q|_R$ is the quotient map for R w.r.t. $\sim$. Hence, if $U \subseteq R/_\sim$, it follows $q^{-1}(U) \cap R$ is open in R if, and only if, U is open in $R/_\sim$. Suppose $U \subseteq R/_\sim$ is a connected open set. If U meets the boundary of R, then we can embed U into $\mathbb{R}^2$ by considering tiles adjacent to R in the plane; call this open set V. Observe $q^{-1}(U) = \bigcup_{w\in\langle f,g\rangle} w(V)$; since multiplication is continuous, it follows $q^{-1}(U)$ is open in $\mathbb{R}^2$. On the other hand, suppose $U \subseteq R/_\sim$ such that $q^{-1}(U)$ is open in $\mathbb{R}^2$. Then, $q^{-1}(U) \cap R$ is open in R, which implies U is open in $R/_\sim$. Thus, q is a quotient map.

Consider $h : \mathbb{R}^2/\langle f,g\rangle \to R/_\sim$ defined by $[x] \mapsto q(x)$ for each $x \in \mathbb{R}^2$. We claim h is well-defined. Suppose $x \sim_{\langle f,g\rangle} y$. Then $x = w(y)$ for some $w \in \langle f,g\rangle$. There exists $w_1 \in \langle f,g\rangle$ such that $w_1(x) \in R$. Suppose $w_1(x) \notin \partial R$. Then, w_1 is unique, and there also exists a unique word $w_2 \in \langle f,g\rangle$ such that $w_2(y) \in R$. Since $w_1 w(y) = w_1(x) \in R$, it follows by uniqueness $w_2 = w_1 w$. On the other hand, if $w_1(x) \in \partial R$, then w_1 need not be unique, but if $w_2 \in \langle f,g\rangle$ such that $w_2(y) \in R$, then $w_1(x) = w_2(y)$ or $w_1(x) = w_3(w_2(y))$ for some $w_3 \in \{f, g, f^{-1}, g^{-1}, fg, f^{-1}g, fg^{-1}, f^{-1}g^{-1}\}$. It follows $w_1(x) \sim w_2(y)$, and therefore $q(x) = q(y)$. Hence, h is well-defined. We also observe h is a bijection. Let $q^* : \mathbb{R}^2 \to \mathbb{R}^2/\langle f,g\rangle$ be the quotient map. Suppose $U \subseteq R/_\sim$ is open. Then, $q^{-1}(U)$ is open in $\mathbb{R}^2$, which implies $q^*\big(q^{-1}(U)\big)$ is open in $\mathbb{R}^2/\langle f,g\rangle$. Indeed $q^*\big(q^{-1}(U)\big) = h^{-1}(U)$, which gives us h is continuous. On the other hand, if $U \subseteq \mathbb{R}^2/\langle f,g\rangle$ is open, then $(q^*)^{-1}(U)$ is open in $\mathbb{R}^2$ and thus $q\Big((q^*)^{-1}(U)\Big) = h(U)$ is open in $R/_\sim$. Thus, h is an open map, and consequently a homeomorphism. Hence, $R/_\sim \cong \mathbb{R}^2/\langle f,g\rangle$. ∎

Theorem 2.3 (Figure 12)**.** $\langle f,g\rangle = \langle f,g \mid [f,g] = 1\rangle$.

Proof. Define $h : \langle f,g\rangle \to \mathbb{Z}^2$ by $w(f,g) \mapsto$ (num f's, num g's). For example, $f^3g^7f^{-3}g^6f \mapsto (1,13)$. Observe h is an isomorphism. ∎

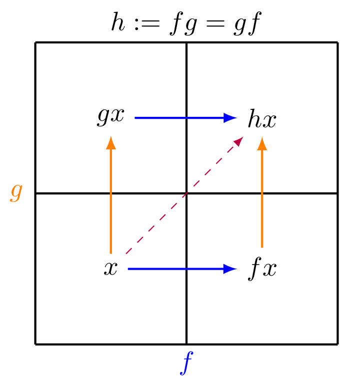


FIGURE 12. f and g commute

Lemma 2.4. *Suppose $G \curvearrowright X$ and G tiles X with tiles of shape T with non-trivial open interior. Then G is discrete.*

Proof. Fix $x \in \text{int}(T)$. To derive a contradiction, suppose G is not discrete. Then, because 1 is not isolated, there exists a sequence of distinct points $\langle g_n \mid n \in \omega \rangle$ converging to 1, implying $\{\text{int}(g_n T) \mid n \in \omega\}$ is a collection of pairwise disjoint sets. But, as we may assume $g_n \neq 1$, $g_n x \in \text{int}(g_n T) \subseteq X \setminus \text{int}(T)$ for each $n \in \omega$. Since $\langle g_n x \mid n \in \omega \rangle$ converges to x, it follows $x \in X \setminus \text{int}(T)$ because $X \setminus \text{int}(T)$ is closed. However, this contradicts the fact $x \in \text{int}(T)$, and we are done. □

We may now consider constructing an octagon from genus 2 surface, as seen in Figure 13. There are 8 sides, so we required an angle of $\frac{2\pi}{8} = \frac{\pi}{4}$. Note we cannot tile $\mathbb{R}^2$ with octagons in a 'nice' way. Figure 14 has the octagon embedded into hyperbolic space.

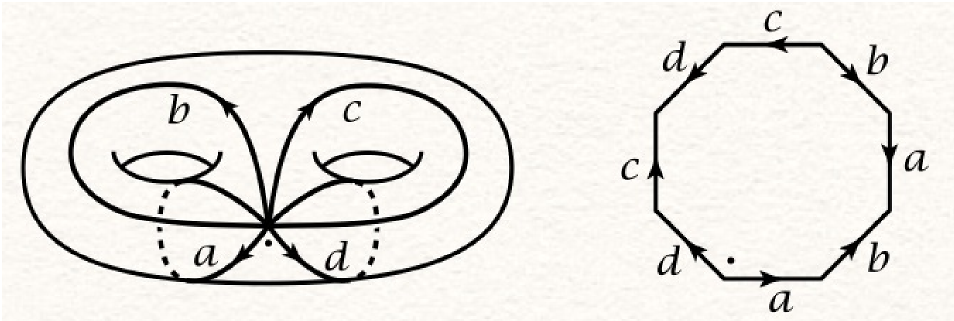


FIGURE 13. Genus 2 to octagon construction

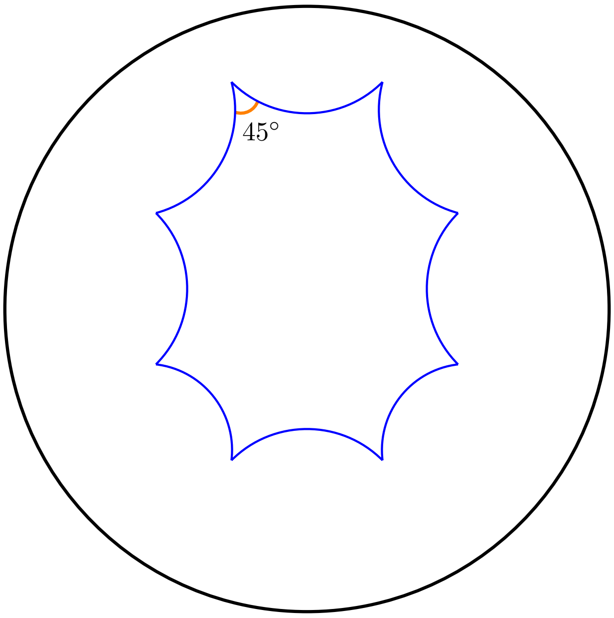


FIGURE 14. Octagon in $\mathbb{H}^2$

2.2. **Hyperbolic Space.** For dimension 2, there are three different types of geometries, as seen in Figure 15. In the elliptic case, there are no parallel lines, with triangle angles sum greater than π, and isometry group SO(3). In the Euclidean case, exactly 1 parallel line, angles sum to π, and isometry group

$$\{M \in \mathrm{GL}\,(2, \mathbb{C}) \mid \mathrm{tr}(M) \in [-2, 2], \det(M) = 1\}.$$

For Hyperbolic geometry, there are infinitely many parallel lines, with triangle angles sum less than π, and isometry group $\mathrm{PSL}\,(2, \mathbb{R})$.

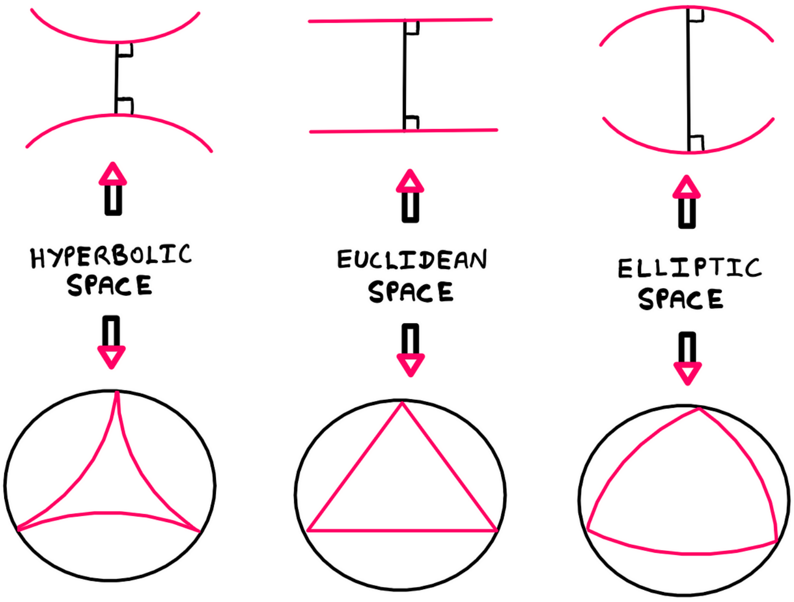


FIGURE 15. Three types of geometry

A map is *conformal*, if it preserves angles. We denote $\mathrm{PSL}\,(2, \mathbb{R})$ by

$$\mathrm{PSL}\,(2, \mathbb{R}) = \{M \in \mathrm{GL}\,(2, \mathbb{R}) \mid \det(M) = 1\}/_\sim,$$

where $M \sim N$ if, and only if, $M = kN$ for some $k \in \mathbb{R}$. Equivalently, $\mathrm{PSL}\,(2, \mathbb{R}) = \mathrm{SL}\,(2, \mathbb{R})/\{\pm I\}$, where I is the 2×2 identity matrix.

On the hyperbolic plane $\mathbb{H}^2 := \{z \in \mathbb{C} \mid \mathrm{im}\,(z) > 0\}$, the metric is given by $d(w, z) = \cosh^{-1}\left(1 + \frac{(z+w)^2}{2\mathrm{im}\,(z)\mathrm{im}\,(w)}\right)$. In other words, $ds^2 = \frac{dx^2+dy^2}{y^2}$.

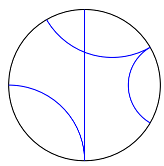


FIGURE 16. Hyperbolic plane using Poincaré disc model

$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \frac{az+b}{cz+d}$ gives translations and rotations. See Figure 17 for circle inversions/Moebius transformation.

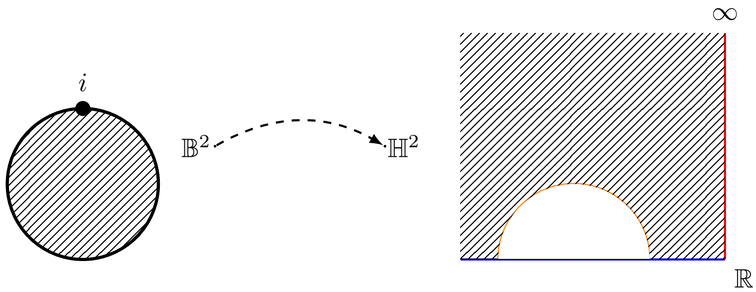


FIGURE 17. Circle inversion

(1) An isometry is called *elliptic*, if it fixes a point in $\mathbb{H}^2$ (corresponds to rotations).
(2) An isometry is called *parabolic*, if a unique point is fixed on boundary, i.e., real line (corresponds to translations).
(3) An isometry is called *hyperbolic*, if there are two fixed points.

Theorem 2.5 (Poincaré, Koebe/Klein Uniformisation Theorem (Conjectured 1883, Proved 1907))**.** *Let S be a surface (Riemann surface, manifold of dimension 2, or a 2D Riemannian manifold with boundary; thought of as a regular surface, but can delete points or discs) with n points and deleted discs. Then,*

(1) *If $2-2g-n>0$, i.e., $\chi(S)=2$, then S is the sphere S^2.*
(2) *If $2-2g-n=0$, i.e., $\chi(S)\in\{0,1\}$, then $S=\mathbb{R}^2/\Gamma$ where Γ is a discrete subgroup of Euclidean isometries* $\operatorname{Isom}(\mathbb{R}^2)$*.*
(3) *If $2-2g-n<0$, i.e., $\chi(S)<0$, then $S=\mathbb{H}^2/\Gamma$, where Γ is a discrete subgroup of* $\operatorname{PSL}(2,\mathbb{R})$*.*

Note. Genus g unwraps to a $4g$-gon on $\mathbb{H}^2$. The surfaces can be sphere one or two punctures, otherwise n-tori.

Only with flat metric: $\mathbb{R}^2/\langle \mathrm{id}\rangle$ (do nothing), $\mathbb{R}^2/\langle f\rangle$ (translation, obtain cylinder), and $\mathbb{R}^2/\langle f,g\rangle$ (obtain torus).

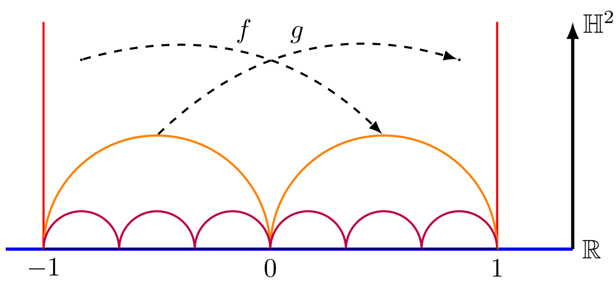


FIGURE 18. Fundamental group punctured torus

Example 2.6. Consider the punctured torus, which has Euler characteristic $2-2-1=-1$. Consider Figure 18 for fundamental group (tiling by quadralateral, so tangent equals 0). We consider Moebius transformations $z\mapsto\frac{az+b}{cz+d}$ where $ad-bc=$

1. Want map to send $-1 \mapsto 0$ and $\infty \mapsto 1$. Hence,

$$0 = \frac{-a+b}{-c+d} \implies a = b$$

and

$$1 = \frac{a \cdot \infty + b}{c \cdot \infty + d} = \frac{a}{c} \implies a = c.$$

Since $ad - bc = 1$, $a = b$ and $a = c$, it follows $ad - a^2 = 1$. Take $a = 1$ and $d = 2$. Let $f := \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$ and $g = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}$.

Recall for dimension two, there are three geometries: spherical, euclidean and hyperbolic. In three dimension, there are 8 possible geometries (8 Thurston geometries, Fundamental groups from their matrices):

- $S^3, \mathbb{R}^3, \mathbb{H}^3$
- $\mathbb{R}^1 \times S^2$, $\mathbb{R}^1 \underset{\sim}{\times \mathbb{H}^2}$
- Nil, Sol, $\mathrm{PSL}\,(2, \mathbb{R})$

Note. The game Hyperbolica is based on $\mathbb{R} \times \mathbb{H}^2$. Another game based on hyperbolic geometry, is that is HyperRogue.

2.3. Triangle Groups.

Theorem 2.7. *If $p, q, r \in \hat{\mathbb{Z}}$, where $\hat{\mathbb{Z}} := \mathbb{Z} \cup \{\infty\}$, and $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} < 1$, then there exists a hyperbolic triangle with angles as in Figure 19.*

Proof. $\frac{\pi}{p} + \frac{\pi}{q} + \frac{\pi}{r} < \pi$. □

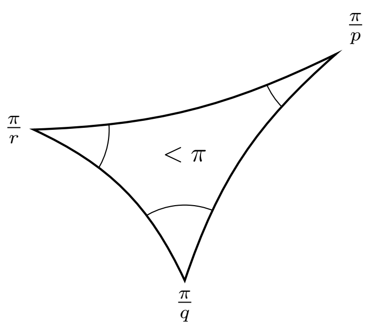


FIGURE 19. Hyperbolic triangle

Can tile the plane by Figure 20, where $\frac{2\pi}{r}$ has q tiles around it and $\frac{\pi}{q}$ has $2r$ tiles around it.

We now consider the group of reflections, as in Figure 21.

Group generated by reflections is the *triangle group*, $G = \langle a, b, c \mid (ab)^r = (bc)^q = (ca)^r \rangle$. Then, $ca = (abbc)^{-1}$.

There are orientation reversing elements, see Figure 22. We therefore consider the *orientation preserving triangle group*, $\tilde{G} = \langle f = ab, g = bc \mid f^p = g^q = (fg)^r = 1 \rangle$, which is generated by those of even length. $G \to \{\text{pres.}, \text{non pres.}\}$, where product of two non-preserving is preserving.

The quotient surface is sphere, seen by its angle. Can have punctures (called an 'orbifold'). Locally triangle paper to make cone - not covered by Euler characteristic.

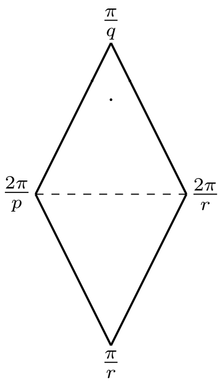


FIGURE 20. Tiles

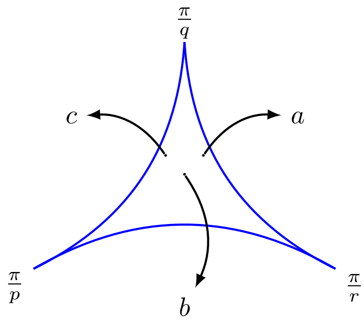


FIGURE 21. Triangle group reflections

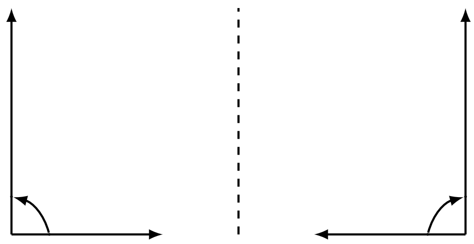


FIGURE 22. Orientation reversing

Example 2.8. Let $p = q = r = \infty$. See Figure 23. Then, we obtain a sphere with four punctures. Let $f = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ and $g = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$. Then, as we have seen before, $\langle f, g \rangle$ is free.

Example 2.9. $\mathrm{PSL}(2, \mathbb{Z})$ is the $(2, 3, \infty)$-triangle group. Let $S = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $T = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, which are elements of $\mathrm{PSL}(2, \mathbb{Z})$. Then, S has infinite order, and T has order 2. Using algorithm/row operations, we show $\mathrm{PSL}(2, \mathbb{Z})$ has the presentation $\left\langle S, T \,\middle|\, S^{\infty} = T^2 = (ST)^3 = 1 \right\rangle$.

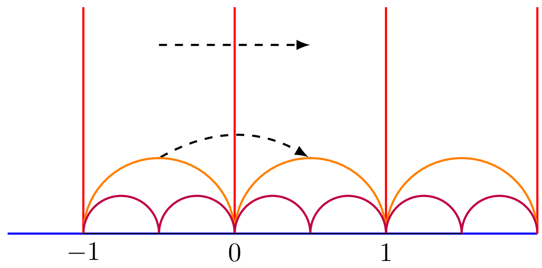


FIGURE 23. $p = q = r = \infty$

Proof. □

2.4. **Bead Groups.** Consider Figure 24, for which $f(\text{ext}\,(A)) \mapsto \text{int}\,(\overline{A})$, $g(\text{ext}\,(B)) \mapsto \text{int}\,(\overline{B})$, $f(A) = \overline{A}$ and $g(B) = \overline{B}$.

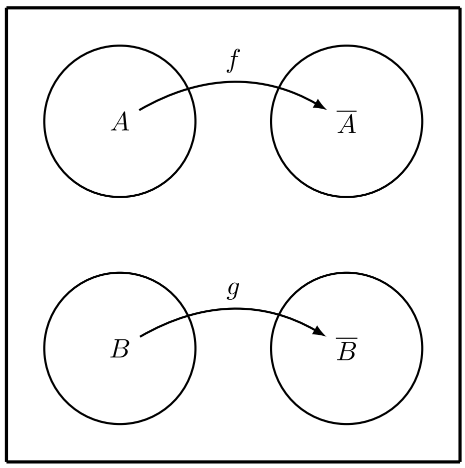


FIGURE 24. f and g for $\mathbb{C}$

Definition 2.10. A *conformal map* preserves angles at all points.

Theorem 2.11. *Every conformal map which is holomorphic $\hat{\mathbb{C}} \to \hat{\mathbb{C}}$, where $\hat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$ is the* Riemann sphere*, and bijective is of the form $z \mapsto \frac{az+b}{cz+d}$ for $a, b, c, d \in \mathbb{C}$ and $ad - bc \neq 0$, that is, a* Möbius transformation.

Note. The proof is by Picard's Little Theorem. Möbius transformations, otherwise called *linear fractional transformations*, have at most 2 fixed points.

$G = \langle f, g \rangle$ is free and also discrete. Discrete since interior minus disc tiles the complex plane, which follows from Lemma 2.4. If discrete, called a *Schottky group*.

What is $\hat{\mathbb{C}}/\langle f, g\rangle$? $\pi : \hat{\mathbb{C}} \to \hat{\mathbb{C}}/\langle f, g\rangle$. A nhood of $\pi(\eta)$: given z can get arbitrarily close to η by applying f. In every nhood of η, there exists $f^n(z)$. So $\pi(\eta)$ lies in every nhood of $\pi(\eta)$ (in particular, not Hausdorff). Define $\Omega(G)$ to be the "limit set" equal to $\overline{\{\text{attractive fixed points}\}}$. $G \curvearrowright X$, G tiles X with tile T. "$X/G = T/\sim$" (Poincaré Polyhedron Theorem). $\Omega/G = T/\sim$. $\pi_1(Q) = G$, $\pi_1\left(\mathbb{T}^2\right) = \mathbb{Z}^2$.

Example 2.12. An example of a discrete, infinitely generated group is that of Accola - atom group, which is discrete since it tiles part of the plane. See Figure 25.

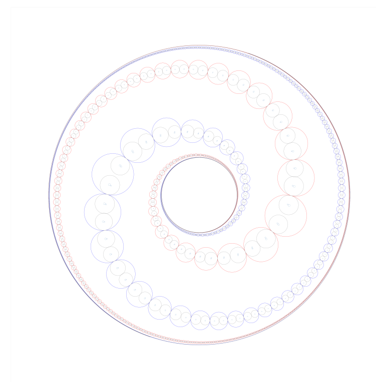


FIGURE 25. Atom group

Theorem 2.13 (Ahlfor's Finiteness Theorem)**.** *If* $G \leq \mathrm{PSL}(2, \mathbb{C})$ *is discrete, finitely generated, and* $\Omega \neq \varnothing$*, then* Ω / G *is a compact surface of finite genus with possibly finitely many punctures.*

Note. Proof uses harmonic analysis on cohomology groups.

3. Trees

3.1. Cayley Graphs.

Definition 3.1. Let G be a group generated by a finite set S. The *Cayley graph of G with respect to S*, denoted by $\mathrm{Cay}(G, S)$, is a directed graph with G as the vertex set, and edge set $\{(g, gs) \mid g \in G, s \in S\}$.

Note. Groups can have more than one Cayley graph. Sometimes we impose $S = S^{-1}$, so that the edges are undirected, and also that $1 \notin S$, so we do not have loops.

Example 3.2. Let $G = (\mathbb{Z}, +)$, $S_1 = \{1, -1\}$, and $S_2 = \{2, 3, -2, -3\}$. Consider $\mathrm{Cay}(G, S_1)$ and $\mathrm{Cay}(G, S_2)$ depicted on the left and right, respectively, of Figure 26.

FIGURE 26. Cayley graphs of $\mathbb{Z}$

Taking different generating sets, say S and S', we obtain different Cayley graphs. Although the graphs are different, they are *quasi-isometric*, an invariant we will discuss later. It is a part of 'coarse geometry (Gromov)', as is the following theorem.

Theorem 3.3 (Gromov Polynomial Growth Theorem)**.** *Every finitely generated group which has polynomial growth is virtually nilpotent.*

Note. A group G is *virtually P*, where P is a group property, if there exists a subgroup H of G with finite index that has property P. Polynomial growth is invariant under choice of generating set, and determines how fast the closed ball around the identity (in Cayley graph) grows with respect to the word metric.

Example 3.4. Let $F_2 = \langle a, b \mid - \rangle$ be the free group of rank 2. Then, Figure 27 depicts the Cayley graph of F_2, where $S = \{a, b\}$. Also, Figure 28 depicts $\mathrm{Cay}\left(\mathbb{Z}^2, \{(1,0),(0,1)\}\right)$.

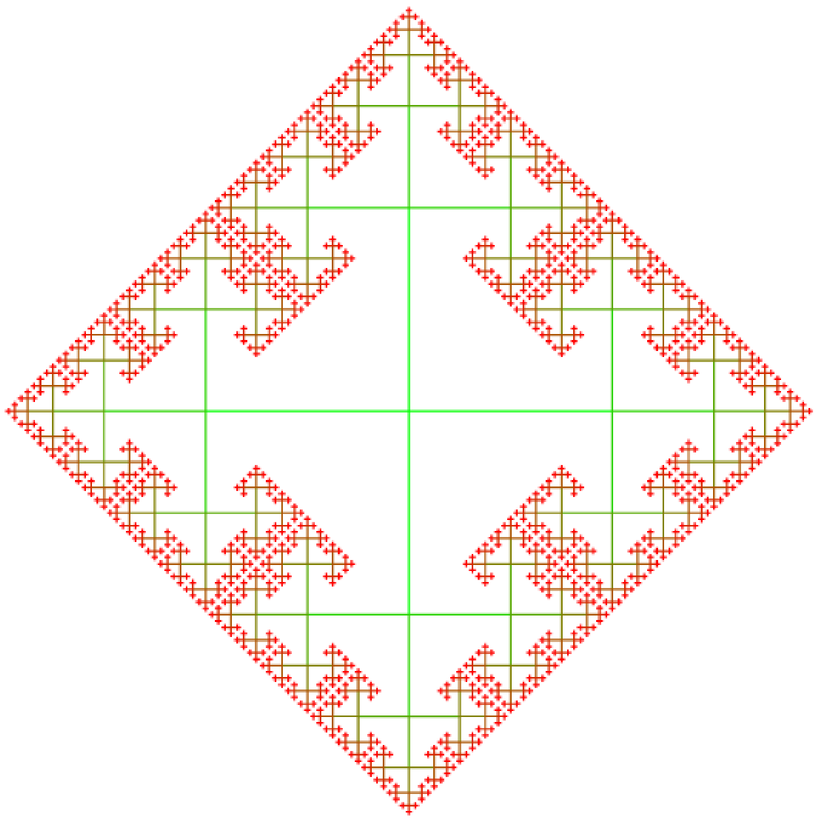


FIGURE 27. Cayley graph of F_2

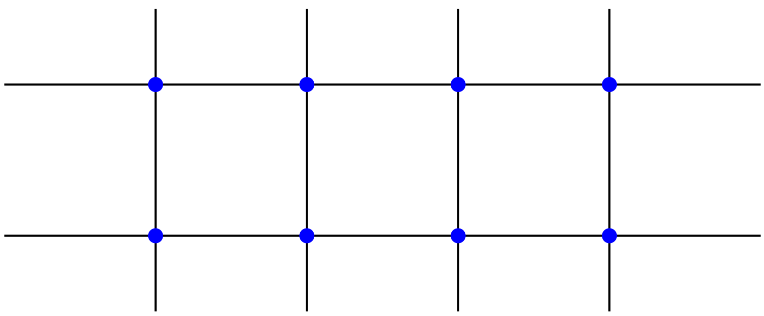


FIGURE 28. Cayley graph of $\mathbb{Z}^2$

3.2. Group Actions on Trees.

Definition 3.5. A *tree* is a connected graph without cycles.

Note. We allow for infinite trees. We will look at *simplicial trees*, and later $\mathbb{R}$-*trees*.

We consider isometries, of which there are two types:

(1) A point is fixed (*elliptic isometry*).
(2) No points are fixed (*hyperbolic isometry*).

Example 3.6. An example of an elliptic isometry is a tree with a vertex for which branches extend symmetrically, such as the central vertex for the Cayley graph of F_2. A rotation around this vertex, mapping the tree to itself, is an elliptic isometry because the central vertex is fixed.

An example of a hyperbolic isometry is a translation along an infinite path in a tree, since it shifts every point along the path by the same distance, and so there are no fixed points. For a concrete example, consider the Cayley graph of $\mathbb{Z}$.

We do not allow edge inversions (see Figure 29, midpoint in blue), otherwise fix midpoint of edge, and can then add the midpoint as a vertex.

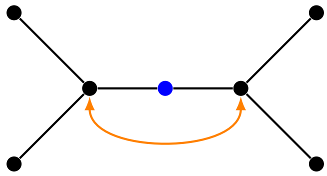


FIGURE 29. Edge inversion

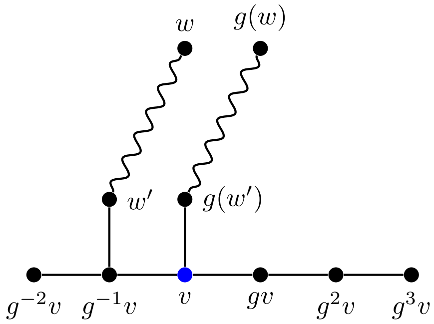


FIGURE 30. Minimal displacement

Let $v \in T$ have minimal displacement, i.e., $\min(g)$. For a hyperbolic element with translation, we call the infinite path in a tree a *translation axis*, or simply an *axis*. For each $n \in \mathbb{N}$, $g^n v$ has minimal displacement because v does. Those vertices not on an axis will have larger displacement.

Proposition 3.7 (See Figure 31)**.** *Let G be a finitely generated group. If $g \in G$ is elliptic, then g fixes the midpoint of (x, gx) for each $x \in T$.*

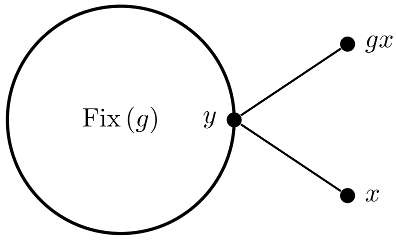


FIGURE 31. Fixes midpoint

Proof. Let $x \in T$. Let n be the distance between $\operatorname{Fix}(g)$ and x. There exists $y \in \operatorname{Fix}(g)$ such that $d(x, y) = n$. Say $v_0 v_1 \dots v_n$ is a path from y to x in T. Now, for $u_i = gv_i$, $u_0 u_1 \dots u_n$ is a path from gy to gx in T. Since g fixes y, it is a path from y to gx in T of length n. It follows $d(gx, y) \leq n$. But also $d(gx, y) \geq n$, otherwise applying a similar argument we obtain $d(x, y) < n$ (a contradiction). Moreover, the path $v_n v_{n-1} \dots v_0 u_1 \dots u_n$ is without backtracking (since otherwise there would be a point in $\operatorname{Fix}(g)$ closer to x than that of y), and so it is geodesic from x to gx. Thus, g fixes the midpoint of (x, gx) for each $x \in T$. $\square$

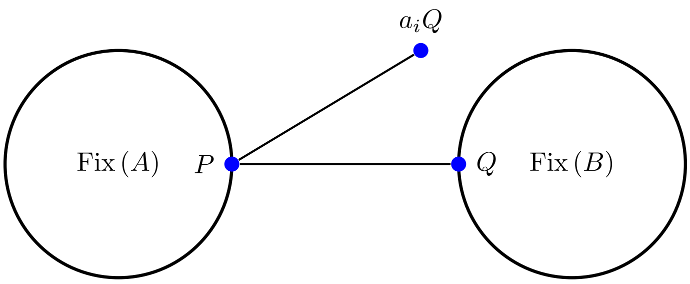


FIGURE 32. Illustration of proof in Lemma 3.8

Lemma 3.8 (See Figure 32)**.** *Let G be a finitely generated group acting on a tree T, with subgroups $A = \langle a_i \mid i \in I \rangle$ and $B = \langle b_j \mid j \in J \rangle$ such that $G = \langle A \cup B \rangle$. Assume $\operatorname{Fix}(A)$ and $\operatorname{Fix}(B)$ are non-empty. If each element of G is elliptic, then $\operatorname{Fix}(A) \cap \operatorname{Fix}(B) \neq \varnothing$.*

Proof. To derive a contradiction, assume $\operatorname{Fix}(A) \cap \operatorname{Fix}(B) = \varnothing$. Let $P \in \operatorname{Fix}(A)$ and $Q \in \operatorname{Fix}(B)$ such that (P, Q) is a geodesic path from $\operatorname{Fix}(A)$ to $\operatorname{Fix}(B)$. Then, $a_i Q = a_i b_j Q$ for each $(i, j) \in I \times J$, since $Q \in \operatorname{Fix}(B)$. Notice there exists $i \in I$ such that $a_i Q \neq Q$, since otherwise $Q \in \operatorname{Fix}(A)$ and we are done. Now, fixing $j \in J$, $a_i b_j$ is elliptic and therefore fixes the midpoint M of $(Q, a_i b_j Q)$. Since $a_i b_j Q = a_i Q$, it follows the midpoint M is independent of $j \in J$. That is to say, $a_i b_j$ fixes M for each $j \in J$. We claim $M = P$. For we observe following the geodesic from Q to P, then from P to $a_i Q$ by applying a_i to the geodesic from Q to P yields a geodesic from Q to $a_i Q$. Indeed, P is the midpoint, so $M = P$. In particular, for each $j \in J$, we get $P = a_i b_j P$ implying $P = a_i^{-1} P = b_j P$. But this implies $P \in \operatorname{Fix}(B)$, a contradiction. Thus, $\operatorname{Fix}(A) \cap \operatorname{Fix}(B) \neq \varnothing$. $\square$

Theorem 3.9 (Serre (Trees, 10, 80))**.** *Let G be a finitely generated acting on a tree T. Suppose that every element of G fixes a point (i.e., each element is elliptic). Then, G has a global fixed point.*

Note. If for every $g \in G$ there exists $x \in T$ such that $gx = x$, then there exists $x \in T$ such that for each $g \in G$ we have $gx = x$.

Proof. Say $G = \langle s_1, \ldots, s_n \rangle$. Let $A_i = \langle s_1, \ldots, s_i \rangle$ and $B_i = \langle s_{i+1} \rangle$ for $i \in \{1, \ldots, n-1\}$. For each $i \in \{1, \ldots, n-1\}$, $\mathrm{Fix}\,(B_i)$ is non-empty because s_{i+1} is elliptic. Note $\mathrm{Fix}\,(A_1)$ is non-empty. By Lemma 3.8, $\mathrm{Fix}\,(A_1) \cap \mathrm{Fix}\,(B_1) \neq \varnothing$. It follows $\mathrm{Fix}\,(A_2)$ is non-empty (since $A_{i+1} = \langle A_i \cup B_i \rangle$), and so again applying Lemma 3.8 we obtain $\mathrm{Fix}\,(A_2) \cap \mathrm{Fix}\,(B_2) \neq \varnothing$. Continuing this way inductively, we obtain $\mathrm{Fix}\,(G)$ is non-empty. ☐

Note. Geodesics in trees are closed paths without backtracking (recall closed paths without backtracking between two points are unique, otherwise can create a cycle and so no longer a tree).

Proposition 3.10 (Tits)**.** *Suppose s is a hyperbolic isometry. Set*

$$m = \inf_{P \in X} \ell(P, sP) \text{ and } T = \{P \in X \mid \ell(P, sP) = m\}.$$

Then,

1. *T is the vertex set of a straight path of X.*
2. *s induces a translation of T of amplitude m.*
3. *Every subtree of X stable under s and s^{-1} contains T.*
4. *If a vertex Q of X is at a distance d from T then $\ell(Q, sQ) = m + 2d$.*

A group Γ acts *freely* on a tree X, if for each $\gamma \in \Gamma \setminus \{1\}$ and $x \in X$, $\gamma x \neq x$. If $\Gamma \curvearrowright X$, then the *quotient graph*, is the graph $\Gamma \setminus X$ with vertices $\{Gx \mid x \in V(X)\}$ and edges $\{Ge \mid e \in E(X)\}$; Gx denotes the orbit of x under G and Ge the orbit of e under G. Note $Gx \sim Gy$ via Ge if, and only if, there exists $a \in Gx$, $b \in Gy$, and $f \in Ge$ such that $f = (a, b) \in E(X)$.

Proposition 3.11. *Suppose $\Gamma \curvearrowright X$, where X is a tree and Γ is a group generated by a hyperbolic element s. Then, Γ acts freely on X, and the quotient graph $\Gamma \setminus X$ contains exactly one circuit, namely $\Gamma \backslash T$; this is a circuit of length m. The injection $\Gamma \setminus T \hookrightarrow \Gamma \setminus X$ is a homotopy equivalence.*

Proof. To derive a contradiction, suppose Γ does not act freely on X. Then, there is $\gamma \in \Gamma \setminus \{1\}$ and $x \in X$ such that $\gamma x = x$. That is to say, there exists a non-trivial elliptic element γ. Since s generates Γ, $\gamma = s^n$ for some $n \in \mathbb{Z}$. We may assume w.l.o.g. $n \geq 1$. We have $s^n x = x$ for some fixed $x \in X$. That is to say, s^n is elliptic. Consider the geodesic paths $(s^i x, s^{i+1} x)$ for each $i \in \{0, \ldots, n-1\}$. The geodesic path (x, sx) is non-trivial, since s is hyperbolic (implying $x \neq sx$). Therefore, the vertices in the geodesic path $(sx, s^2 x)$ are distinct from one another and differ from those in (x, sx) (excluding the common endpoint sx). Distinct from one another because $sy = sz$ implies $y = z$. Different from those in (x, sx) because if we had $sy = z$ for some y, z in (x, sx), to avoid backtracking we must have $s^2 x = x$. But this is just an inversion, which are not allowed. Continuing this way, the same, but this means $s^n x \neq x$, a contradiction.

Note $Gx = \{s^n x \mid n \in \mathbb{Z}\}$ for each $x \in T$. Now, we observe $\Gamma \setminus T$ is a circuit. For we let $x \in T$, and $x_0 \ldots x_m$ be a path from x to sx. Then, since m is infimum, $Gx_0, \ldots, Gx_{m-1}$ are distinct vertices of $\Gamma \setminus T$, and are pairwise adjacent. Moreover, $Gx_0 = Gx_m$ because $x_m = sx \in Gx_0$. Therefore, $Gx_0, \ldots, Gx_m$ forms a circuit and are the vertices of $\Gamma \setminus T$. It remains to show uniqueness. To derive a contradiction, suppose there is another circuit. ☐

Lemma 3.12. *Let $X_1, \ldots, X_m$ be subtrees of a tree X. If the X_i meet pairwise, then their intersection is non-empty.*

Proof. ☐

The "geometry at infinity" for trees is that of *ends of trees*, which are equivalence classes of rays. A *ray* is a copy of the half-line. The equivalence relation on rays $\sim$ is defined by $r_1 \sim r_2$ if, and only if, r_1 and r_2 eventually coincide.

Proposition 3.13. *Suppose $G \curvearrowright T$, where T is a tree and G is a group. Suppose each element of G is elliptic, but that $\operatorname{Fix}(G) = \varnothing$. If $x \in T$, let $g \in G$ be such that $gx \neq x$ and let x_1 be the vertex of the geodesic (x, gx) at distance 1 from x. Then, x does not depend on the choice of g. Let $f : T \to T$ be the map $x \mapsto x_1$; one has $f \circ g = g \circ f$ for each $g \in G$. Then, for each $x \in T$, the sequence $\langle f^n x \mid n \in \mathbb{N} \rangle$ converges to an end of T which is independent of x and fixed by G.*

Proof. Take the subtrees to be $\operatorname{Fix}(g)$ for each $g \in G$. □

Local field:

- $\mathrm{PSL}(2, \mathbb{R})$ acts on $\mathbb{H}^2$
- $\mathrm{PSL}(2, \mathbb{C})$ acts on $\mathbb{H}^3$
- non-archimedean and p-adic $\mathbb{Q}_p$ correspond to $\mathrm{PSL}(2, \kappa)$
- Lawson surface and $\mathrm{PSL}(2, \kappa)$ acts on Bruhat-Tits tree.

Let A and B be two hyperbolic elements (translations, translation axis, no fixed elements) such that $\mathrm{Ax}(A) \cap \mathrm{Ax}(B) = \varnothing$.

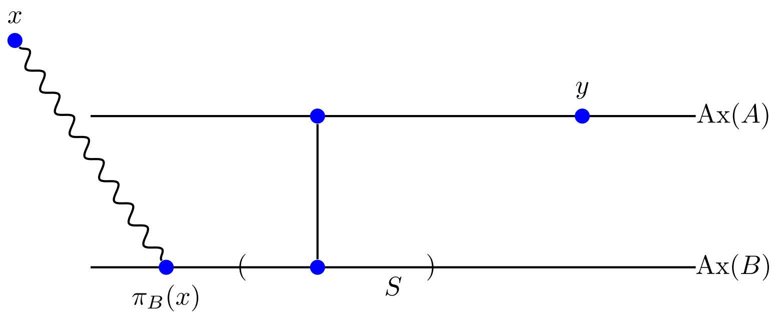


Figure 33. Disjoint axis

Proposition 3.14 (Figure 33)**.** *$\langle A, B \rangle$ is a free group.*

Proof. Let π_B be the projection onto $\mathrm{Ax}(B)$. For $x \in T$, π_B is the unique closest element on $\mathrm{Ax}(B)$ to x. Unique connected element by a path exists because connected, unique because otherwise obtain a cycle (and we have a tree). Let S be open (not including endpoints) of length $\ell(B)$. Let $X_1 = \pi_B^{-1}(S)$ and $X_2 = T \setminus \pi_B^{-1}(S)$. Consider $y \in X_1$, which is hit with B. Then, $By \in X_2$. □

Question 3.15. What if the axes overlap? For example, $\ell(A) = 2$ and $\ell(B) = 3$.

Towards Nielsen-Scheier:

Proposition 3.16. *Any connected graph Γ has a maximal subtree.*

Proof. Look at set of all trees in Γ. Partially order them by inclusion. Every chain has a maximal element $\bigcup_i T_i$. Zorn's Lemma guarentees there is a maximal element. □

Claim: In a connected graph a maximal tree contains all vertices.

We now go over quotient graph and tree of representatives (where the latter is key towards Nielsen-Schreier).

Tree of representatives: lift of a maximal subtree. $G \curvearrowright X$, G group and X tree, without inversions. Edges are directed where if e is an edge, $o(e)$ is the outgoing vertex and $t(e)$ is terminal vertex. Quotient graph: denoted $G \setminus X$, which is a graph. Recall every graph has a maximal subtree.

Proposition 3.17. *Every subtree T of $G \setminus X$ lifts to a subtree of X.*

Proof. Let $\mathcal{T}$ be family of trees in X whose projection into $G \setminus X$ is injective. Let T' be a maximal element in $\mathcal{T}$ (order by inclusion, maximal by Zorn's Lemma). Assume for contradiction that $p(T') \neq T$, where $p : X \to G \setminus X$ is projection. Then there exists an edge in T which is not in $P(T')$, may assume $o(e) \in P(T')$. Note $t(e) \notin P(T')$, otherwise can build a circuit. Then you can lift up to top to construct a larger tree. How: lift e to an edge $\tilde{e}$ in X. May assume $o(\tilde{e})$ in T, then can add $\tilde{e}$ and $t(\tilde{e})$ to T' to find a tree strictly containing T'. □

Theorem 3.18. *A group is free if, and only if, it acts freely on a tree.*

Proof. Suppose G is free. Then, acts freely on a tree, as discussed (acts freely on its Cayley graph).

Conversely, suppose G acts freely on a tree X. Let T be a tree of representatives of $X \mod G$. Then if

$$S = \{1 \neq g \in G \mid \exists y \in E_+, o(y) \in T, t(y) \in gT\}.$$

This is orientation preserving; also, we have edges E_+ and E_-, which are directed.

Remark: gT is set of translates of T. These are pairwise disjoint: $gt_1 = ht_2$ implies $h^{-1}gt_1 = t_2$, so these project to same element in quotient graph, implying $p(t_1) = p(t_2) = t$, and therefore $h^{-1}gt = t$ implying $h = g$ since it acts freely.

Simultaneous contraction of the gT's (results in a tree (gT)). $(gT) \to \operatorname{Cay}(G, S)$, where S is as above (positive edges connecting T to gT). Goal: show they are isomorphic. As (gT) is a tree, will follow $\operatorname{Cay}(G, S)$ is a tree, so G will be freely generated by S. Edges in $\operatorname{Cay}(G, S)$ are (g, gs). Vertices: $gT \mapsto g$. Edges: $\langle gT, hT\rangle \mapsto (g, s)$ where $s = g^{-1}h \in S$. It is clear the map is well-defined, injective and surjective. □

We now obtain Nielsen-Schreier: Every subgroup of a free group is free. With some more work:

- Schreier-Index Formula.
- F_2 contains F_r for every $r \geq 2$.

4. Free groups

Let S be a set. We consider words in S, for example, $w = abc^{-2}a$ where $a, b, c \in S$. Reduced words are those removing instances of aa^{-1} and $a^{-1}a$; for example,

$$w = a^7a^{-2}ba^{-1}bb^2 = a^5ba^{-1}b^3.$$

We have an equivalence of words, where $w \sim w'$ if one can be obtained from the other by 'scratching out' or inserting aa^{-1}.

Proposition 4.1. *Every word is equivalent to a unique reduced word.*

For a free group, elements are equivalence classes of words (reduced words), with binary operation of concatenation. Take $w_1 = aba$ and $w_2 = a^{-1}b^2$. Then, $w_1 * w_2 = abaa^{-1}b^2 = ab^3$.

Proposition 4.2. *A map from a set S to a group H can be extended uniquely to a homomorphism $F(S) \to H$.*

Note. Simply define ϕ by $\phi(s_1s_2) = \phi(s_1)\phi(s_2)$ and $\phi(s^{-1}) = \phi(s)^{-1}$. As there are no relations, this is well-defined.

Corollary 4.3. *Every group is a quotient of a free group.*

Proof. Take the inclusion map $G \hookrightarrow G$, and extend uniquely to a homomorphism $\phi : F(G) \to G$. Then, $G \cong F(G)/\ker(\phi)$ by the First Isomorphism Theorem. □

Question 4.4. Is every subgroup of a free group free?

The answer to the above question is yes, and is the Nielsen-Schreier Theorem. Several proofs, including algebraic proofs, and geometric/topological proofs (groups acting on trees).

If we consider $F(x, y, z)$, and $H = \langle x, y, xy \rangle$, by removing xy we obtain $H = \langle x, y \rangle$. By doing this, we now see H is free - less obvious with other examples.

We now have a key proposition, from which we will deduce Nielson-Schreier.

Proposition 4.5. *A group is free if, and only if, it acts freely on a tree.*

Note. A group *acts freely* on a tree, if there are no inversions (i.e., no flipping edges), and for each $x \in T$, $gx = x$ implies $g = 1$.

Question 4.6. Given a free group, find a tree on which G on which G acts freely?

Recall finitely generated group $G = \langle S \rangle$, $\mathrm{Cay}(G, S)$ where vertices are elements of G and edges (g, gs) for $s \in S$.

(1) $\mathrm{Cay}(G, S)$ is connected if, and only if, S generates G. For if S generates G, then there is a path from the identity element to every $g \in G$, and thus a path between every $g, h \in G$. On the other hand, if $\mathrm{Cay}(G, S)$ is connected, there is a path from the identity element to each $g \in G$, where we may represent g as the product of elements of S.
(2) $\mathrm{Cay}(G, S)$ is a tree if, and only if, S freely generates G. For if S freely generates G, $\mathrm{Cay}(G, S)$ is connected by (1), but also there cannot be any cycles (otherwise obtain a relation). On the other hand, if $\mathrm{Cay}(G, S)$ is a tree, then S generates G by (1), and it is free because a relation induces a cycle.

If F is freely generated by S, then the *rank* of F is $|S|$.

Theorem 4.7. *$F(X) \cong F(Y)$ if, and only if, $|X| = |Y|$.*

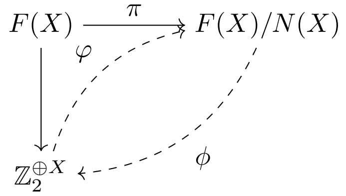
Proof. Suppose $|X| = |Y|$. There exists a bijection from X to Y, which can be uniquely extended; can embed X into $F(Y)$ via the bijection with Y. $\alpha : F(X) \to F(Y)$, $\beta : F(Y) \to F(X)$ where $\alpha \circ \beta = \mathrm{id}_{F(Y)}$ and $\beta \circ \alpha = \mathrm{id}_{F(X)}$.

Conversely, suppose $F(X) \cong F(Y)$. Let $N(X)$ be the squares in $F(X)$. This is a normal subgroup of $F(X)$. For if $n^2 \in N(x)$, $g \in F(X)$, then $g^{-1}n^2g = \left(g^{-1}ng\right)^2$. Hence,

$$F(X)/N(X) \cong F(Y)/N(X).$$

Tales squares to squares. Goal: figure out what $F(X)/N(X)$ is?

We claim $F(X)/N(X) \cong \mathbb{Z}_2^{\oplus X}$. Consider the following diagram, where $F(X)$ mapping to $\mathbb{Z}_2^{\oplus X}$ is given by mapping generator x_i to ith coordinate of $\mathbb{Z}_2^{\oplus X}$.



Now, for our claim, we firstly show $F(X)/N(X)$ is Abelian. Observe

$$ghg^{-1}h^{-1} = \pi(g')\pi(h')\pi\left({g'}^{-1}\right)\pi\left({h'}^{-1}\right) = \pi(g'h')^2 = 1,$$

since $\pi(g') = \pi\left(g'^{-1}\right)\pi\left(g'^{2}\right)$, where $\pi\left(g'^{2}\right) = 1$ (squares sent to trivial). Hence, $F(X)/N(X)$ is indeed Abelian.

Now, by our claim, $\mathbb{Z}_2^{\oplus X} \cong \mathbb{Z}_2^{\oplus Y}$, which are vector spaces over $\mathbb{Z}_2$ of equal dimension, and thus $|X| = |Y|$. □

Note. Finite case: Steinitz exchange.

5. Coarse Geometry

Throughout, let (X, d) be a metric space. We now go over length spaces and geodesic spaces (part of metric geometry).

Definition 5.1. A *curve* is a continuous map $\sigma : [0,1] \to X$. Define

$$L(\sigma) := \sup_k \sum_{i=1}^{k} d(\sigma(t_{i-1}), \sigma(t_i))$$

over all subdivisions t_i of $[0,1]$. A curve σ is a *rectifiable curve*, if $L(\sigma) < \infty$.

Let $x, y \in X$. Define $d_i(x,y) = \inf L(\sigma)$ over all rectifiable curves $\sigma : [0,1] \to X$ from x to y (inner or length metric). If no rectifiable curves exist from x to y, $d_i(x,y) = \infty$. We say X is a *length space*, if $d_i = d$.

Note. $d_i(x,y) \geq d(x,y)$.

A space is *geodesic*, if there is a geodesic between x and y for each $x, y \in X$. A geodesic is a curve between x and y 'travelled proportionally'.

Note. Geodesic spaces are length spaces. Converse is not true. Consider $\mathbb{R}^2 \setminus \{(0,0)\}$, which is not geodesic (take two antipodal points on unit circle). However, it is a length space. See Figure 34.

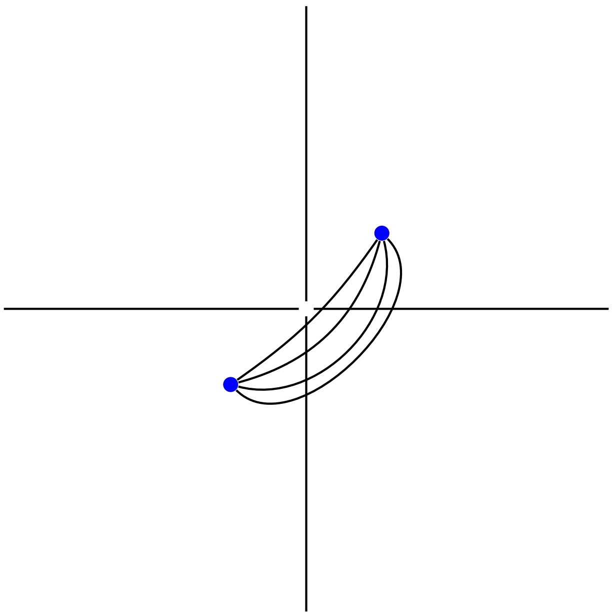


Figure 34. $\mathbb{R}^2 \setminus \{(0,0)\}$

Natural question: when is a length space a geodesic space?

Theorem 5.2 (Hopf-Rinow, Cohn-Vossen, 1935)**.** *Let X be a length space. If X is complete and locally compact, then*

(1) *X is proper, that is, every closed bounded subset is compact.*
(2) *X is a geodesic space.*

Lemma 5.3 (Arzela Ascoli)**.** *If X is a compact metric space, Y is a separable metric space, then every equicontinuous sequence of maps $f_i : Y \to X$ has a subsequence that converges uniformly on compact subsets to a continuous map $f : Y \to X$.*

Proof. Let $Q = \{q_i \mid i \in \mathbb{N}\}$ be a countable dense subset of Y. $f_i(q_1)$; pick a subsequence $(f_{1,i})$ of (f_i) such that $(f_{1,i}(q_1))$ converges (can do this because of sequential compactness). Pick a subsequence $(f_{2,i})$ of $(f_{1,i})$ such that $(f_{2,i}(q_2))$ converges. Continue in this fashion. $(f_{i,i})$ converges pointwise on Q to a map $f : Q \to X$. But we need $f : Y \to X$. For every $\epsilon > 0$, there exists $\delta > 0$ such that if $d(y, y') < \delta$, then $d(f_n(y), f_n(y')) < \epsilon$ for all n. Take the limit: $d(f(q), f(q')) \leq \epsilon$ for each $q, q' \in Q$. There is a unique continuous extension of f to all of Y, since X is complete.

Remains to prove: converges uniformly on compact subsets. We do so by employing "$\frac{\epsilon}{3}$ argument". For $C \subseteq Y$, fix $N > 0$ such that for all $y \in C$, there exists $j(y) < N$ with $d(y, q_j(y)) < \delta$ (follows from compactness by looking at open balls of radius δ). Pick M sufficiently large such that $d(f_{i,i}(q_j), f(q_j)) < \epsilon$ for all $i > M$ and $j < N$. Then,

$$d(f(y), f_{i,i}(y)) \leq d(f(y), f(q_j(y))) + d(f(q_j(y)), f_{i,i}(q_j(y))) + d(f_{i,i}(q_j(y)), f_{i,i}(y)) < 3\epsilon.$$

□

We now consider CAT(0) spaces, which corresponds to symmetric spaces and buildings (Euclidean). CAT stands for Cartan-Alexandrov-Topogrov. CAT(0) spaces are those of non-positive curvature.

Definition 5.4. Let (X, d) be a metric space. If we consider the triangle Δ of $x, y, z \in X$, then its *comparison triangle*, in $\mathbb{E}^2$ is a triangle Δ' of points $x', y', z' \in \mathbb{E}^2$ such that $d(x, y) = d_{\mathbb{E}^2}(x', y')$, $d(x, z) = d_{\mathbb{E}^2}(x', z')$ and $d(y, z) = d_{\mathbb{E}^2}(y', z')$. Then, (X, d) is a *CAT(0) space*, if the distance between points on Δ are less than or equal to the distance of corresponding points on Δ' (e.g., $d(x, t) \leq d(x', t')$, see Figure 35).

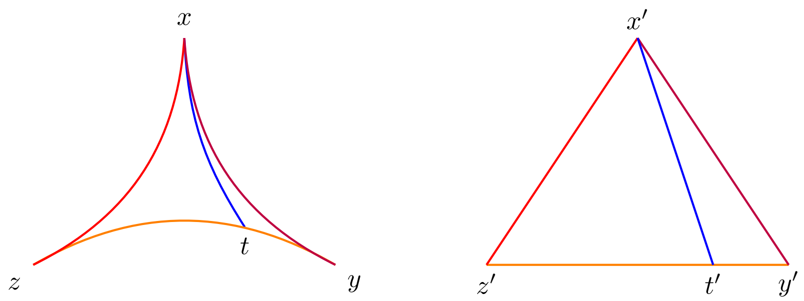


FIGURE 35. Comparison triangle in $\mathbb{E}^2$

We now consider the Bruhat-Tits Fixed Point Theorem. History: Elie Cartan (1928) Simply connected, complete Riemannian manifold of non-positive section curvature. Then, every finite set has a center, i.e., the map $x \mapsto \sum_{i=1}^{n} d(x, x_i)^2$ for a fixed n-tuple $x_1, \ldots, x_n$ has a minimum. That is, the centroid, $\frac{\sum}{n}$.

Francois Bruhat-Jaques Tits: Same is true for Euclidean buildings.

Definition 5.5. The *circumradius* (short: radius) of a bounded set Y in (X, d), is

$$r(Y) = \inf_{y \in \overline{Y}} \left(\sup_{a \in Y} d(x, a) \right) = \inf \left\{ r \in \mathbb{R}^+ \;\middle|\; Y \subseteq \overline{B_r(x)}, x \in X \right\}.$$

Claim: Existence of a center. Let X be a complete CAT(0) Space and Y a bounded subset of X. Then there exists a unique $p \in X$ such that $\overline{B_{r(Y)}(p)}$ contains Y.

Suppose we have proved this claim. Then, we can deduce the Bruhat-Tits Fixed Point Theorem (from claim, circumcenter same, so center fixed).

Theorem 5.6 (Bruhat-Tits Fixed Point Theorem)**.** *Let $G \leq \mathrm{Isom}(X)$, X a complete* CAT(0) *space. If G stabilises a bounded subset Y (e.g., if G is finite), then G fixes a point of X (i.e., $X^G \neq \varnothing$) and X^G is convex.*

Proof. See Figure 36. We see $p_t = (1-t)x + ty$, so

$$d^2(z, p_t) \leq (1-t)d^2(z, x) + td^2(z, y) - t(1-t)d^2(x, y).$$

Let us prove our claim from before. We first show uniqueness of a center, so assume a and b are centers. Put $r(x, Y) = \sup_{y \in Y} d(x, y)$. We have

$$d^2(z, m) \leq d^2_{\mathbb{E}^2}(z', m'),$$

and after some work get

$$d^2(a, b) \leq 2\big(r^2(a, Y) + r^2(b, Y)\big) - 4r^2(Y).$$

If a, b are both center's, then $r(a, Y) = r(b, Y) = r(Y)$. So, $d^2(a, b) \leq 0$ implying $a = b$, obtaining uniqueness. As for existence, construct a sequence of points (x_n) in X such that $r(x_n, Y) \to r(Y)$. Take $a = x_m$ and $b = x_n$ for sufficiently large m, n. Then, (x_n) is a Cauchy sequence, since

$$2\big(r^2(a, Y) + r^2(b, Y)\big) - 4r^2(Y) \to 0.$$

By completeness, $(x_m) \to p$. □

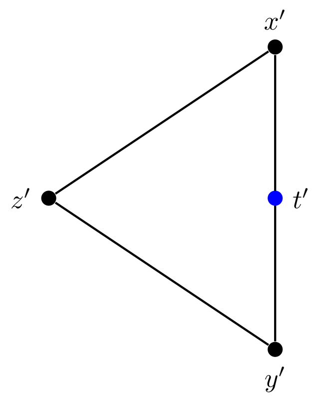


FIGURE 36. Bruhat-Tits Fixed Point Theorem

Theorem 5.7 (Hopf-Rinow, Cohn-Vossen, 1935)**.** *Let X be a length space. If X is complete and locally compact, then*

(1) *X is proper, that is, every closed, bounded set of X is compact.*
(2) *X is a geodesic space.*

Proof. For (1), suffices to prove it for closed balls $B[a,r]$. Let I be the interval of radii's such that it is true. We prove I is open and closed. To this end, we now show I is open. Suppose $r \in I$. We can cover $B[a,r]$ with finitely many open balls $B(x_i, \epsilon_i)$, since $r \in I$ (and thus $B[a,r]$ is compact). Observe

$$\left(X \setminus \left(\bigcup_{i=1}^{n} B(x_i, \epsilon_i)\right)\right) \cap B[a,r] = \varnothing,$$

where $X \setminus (\bigcup_{i=1}^{n} B(x_i, \epsilon_i))$ is closed and $B[a,r]$ is closed and compact. Therefore, there is a minimal distance between them, say 2δ. Therefore, $B[a, r+\delta] \subseteq \bigcup_{i=1}^{n} B[x_i, \epsilon_i]$. Now, we show I is closed. Suppose $[0,\rho) \subseteq I$. We wish to show $\rho \in I$. Suffices to prove: Every sequence of points such that $d(a, x_n)$ converges to ρ has a convergent subsequence (sequential compactness equivalent to compactness). Let ϵ_p be a sequence of positive numbers tending to zero. For each p and n, find y_n^p such that $d(a, y_n^p) \le \rho - \epsilon_{\frac{p}{2}}$; $d(y_n^p, x_n) \le \epsilon_p$ (follows from being a length space). From each p, can extract a convergent subsequence $y_n^p \in B[a, \rho - \epsilon_{\frac{p}{2}}]$ for $n \in \mathbb{N}$ using diagonal argument (similar to Arzela-Ascoli).

Diagonal argument: find $(n_k)_{k\in\mathbb{N}}$ such that $\left(y_{n_k}^p\right)$ converges for all p. Claim: $(x_{n_k})_{k\in\mathbb{N}}$ is a Cauchy sequence. For every $\epsilon > 0$, there exists $M \in \mathbb{N}$ such that if $m, n \ge M$, then $d(x_n, x_m) < \epsilon$ (we omitted the k's in the subscript). Choose p such that $\epsilon_p < \frac{\epsilon}{3}$. Choose n, m large enough such that $d(y_n, y_m) < \frac{\epsilon}{3}$. Then, $d(x_n, x_m) < \epsilon$ by "$\frac{\epsilon}{3}$ argument". This comples proof of (1), i.e., X is proper.

We now show X is a geodesic space. Length space: paths c_n such that $\ell(c_n) \le d(a,b) + \frac{1}{n}$. Claim: such a family of paths is equicontinuous. For every $t, t' \in [0,1]$,

$$|t - t'| = \frac{\ell\left(c_n|_{[t,t']}\right)}{\ell(c_n)} \ge \frac{d(c_n(t), c_n(t'))}{d(a,b)+1}.$$

The image of these paths is contained in a closed bounded set, which is compact by (1). By Arzela-Ascoli, this converges uniformly to a map $c : [0,1] \to X$. $\ell(C) \le \liminf \ell(c_n) = d(a,b)$. □

Definition 5.8. A group G acts *properly* on X, if for each $x \in X$ there exists a nhood U of x such that $g(U) \cap U \ne \varnothing$ for only finitely many $g \in G$. A group G acts *cocompactly*, if there exists compact $C \subseteq X$ such that $GC = X$. We say G acts *geometrically* on X, if G acts both properly and cocompactly on X.

Definition 5.9. We say $f : (X, d_X) \to (Y, d_Y)$ is a *quasi-isometric embedding*, if there exists $A \ge 1$, $B \ge 0$ such that

$$\frac{1}{A} d(x_1, x_2) - B \le d(f(x_1), f(x_2)) \le A d(x_1, x_2) + B.$$

It is a *quasi-isometry*, if in addition there exists $C \ge 0$ such that for each $y \in Y$, there exists $x \in X$ such that $d(f(x), y) \le C$.

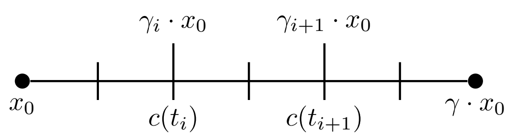


FIGURE 37. Svarc-Milnor

Theorem 5.10 (Svarc-Milnor)**.** *Let X be a length space. If Γ acts properly and cocompactly (i.e., geometrically) by isometries on X, then Γ is finitely generated, and for any choice of base point $x_0 \in X$, the map $\gamma \mapsto \gamma \cdot x_0$ is a quasi-isometry.*

Proof. Let C be compact such that $\Gamma \cdot C = X$. Choose $x_0 \in X$, and $D > 0$ such that $C \subseteq B\big(x_0, \frac{D}{3}\big)$. Let $A = \{\gamma \in \Gamma \mid \gamma \cdot B(x_0, D) \cap B(x_0, D) \neq \varnothing\}$. Claim: A is finite. To derive a contradiction, suppose A is infinite. By Hopf-Rinow Theorem, $B[x_0, D]$ is compact. Fix $x \in B[x_0, D]$. There exists nhood U_x of x such that $\gamma U_x \cap U_x \neq \varnothing$ for only finitely many $\gamma \in \Gamma$. Then, $V_x := U_x \cap B[x_0, D]$ is an nhood of x in $B[x_0, D]$. As $B[x_0, D]$ is compact, there are finitely many $x_1, \ldots, x_n$ such that $\{V_{x_i} \mid i \in \{1, \ldots, n\}\}$ covers $B[x_0, D]$. Of course, $\gamma \cdot V_{x_i} \cap V_{x_i} \neq \varnothing$ for finitely many $\gamma \in \Gamma$. As A is infinite, there exists i and j such that $\gamma \cdot V_{x_i} \cap V_{x_j} \neq \varnothing$ for infinitely many $\gamma \in \Gamma$. Now, we claim there are infinitely many $\gamma_1 \neq \gamma_2 \in \Gamma$ such that $\gamma_1 \cdot V_{x_i} \cap \gamma_2 \cdot V_{x_i} \neq \varnothing$. Once we show this, it implies $\gamma \cdot V_{x_i} \cap V_{x_i} \neq \varnothing$ for infinitely many $\gamma \in \Gamma$, a contradiction (and thus it follows A is finite).

Claim: $\Gamma = \langle A \rangle$. Let $\gamma \in \Gamma$. Consider the geodesic from x_0 to $\gamma \cdot x_0$. Partition $0 = t_0 < t_1 < \ldots < t_n = 1$ (see Figure 37). Partition such that $d(c(t_i), c(t_{i+1})) \leq \frac{D}{3}$. Choose $\gamma_i \in \Gamma$ such that $d(c(t_i), \gamma_i \cdot x_0) \leq \frac{D}{3}$. Distance between $d(\gamma_i x_0, \gamma_{i+1} x_0) \leq D$. Hence, $\gamma_i^{-1} \gamma_{i+1} \in A$, since isometry. Proves claim $\Gamma = \langle A \rangle$.

To prove quasi-isometry: $\frac{d(x_0, \gamma x_0)}{\frac{D}{3} + 1}$. Then $d_A(1, \gamma)$ is the number of elements in partition. □

Definition 5.11. Let (X, d) be a metric space, and $\gamma \in \text{Isom}(X)$. The *displacement function*, $d_\gamma : X \to \mathbb{R}^+$ is defined by $d_\gamma(x) = d(x, \gamma(x))$. The *translation length*, is $|\gamma| := \inf \{d_\gamma(x) \mid x \in X\}$. The *min-set*, which could be empty, is

$$\min(\gamma) := \{x \in X \mid d_\gamma(x) = |\gamma|\}$$

and

$$\min(\Gamma) := \bigcap_{\gamma \in \Gamma} \min(\gamma).$$

Example 5.12. $f_\lambda : \mathbb{R}^n \to \mathbb{R}^n$, defined by $x \mapsto x + \lambda$ for some $\lambda \in \mathbb{R}^n$. Then, $|f_\lambda| = \|\lambda\|$, where $\min(f_\lambda) = \mathbb{R}^n$.

Isometries of X fall into 3 classes, where the first two are often called semi-simple:

- *Elliptic*, if $|\gamma| = 0$ and $\min(\gamma) \neq \varnothing$.
- *Hyperbolic*, if $|\gamma| > 0$ and $\min(\gamma) \neq \varnothing$.
- *Parabolic*, if $\min(\gamma) = \varnothing$.

An example we will look at later, is that of the hyperbolic metric, $\frac{dy}{y^2}$, where $z \mapsto z + 1$.

$\mathbb{R}$-trees: group acting freely on $\mathbb{R}$-trees need not be free, unlike simplicial trees we have seen.

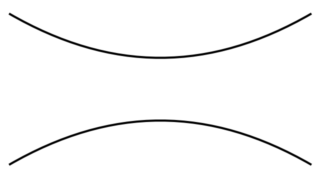


FIGURE 38. Convexity of distance function

Properies of Min-Sets include:

(1) $\min(\gamma)$ is γ-invariant, and $\min(\Gamma)$ is Γ-invariant.

(2) $\alpha \in \mathrm{Isom}(X)$, then $|\gamma| = \left|\alpha\gamma\alpha^{-1}\right|$.
(3) X CAT(0) implies d_γ is convex and $\min(\gamma)$ is a closed convex set. Can prove convexity of the distance function, that is, for two geodesics c_1, c_2:

$$d(c_1(t), c_2(t)) \leq (1-t)d(c_1(0), c_2(0)) + td(c_1(1), c_2(1)).$$

See Figure 38.
(4) If $C \subseteq X$ is non-empty, complete, convex and $\gamma \cdot C = C$, then
- $|\gamma| = |\gamma|_C$;
- γ is semi-simple if, and only if, $\gamma|_C$ is semi-simple.

Projection in CAT(0)-spaces: $p(x)$ unique closest point in C to x. Has the property that angle $\geq \frac{\pi}{2}$ when comparing to all other points. Also, $p(\gamma \cdot x) = \gamma \cdot p(x)$ and $d(\gamma x, x) \geq d(\gamma \cdot p(x), p(x))$ for each $x \in X$. See Figure 39.

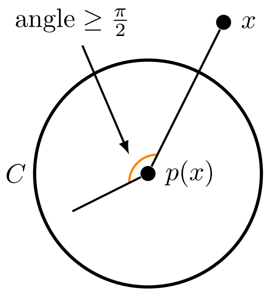


FIGURE 39. Projection in CAT(0)-Spaces

Hyperbolic plane: $\mathbb{H}^2 = \{x + iy \mid y > 0\} \subseteq \mathbb{C}$, where $ds^2 = \frac{dx^2+dy^2}{y^2}$. Action by Moebius transformations. We have $f \in \mathrm{SL}(2, \mathbb{R})$ is

- elliptic iff diagonalisable in $\mathrm{SL}(2, \mathbb{C})$ but not in $\mathrm{SL}(2, \mathbb{R})$;
- hyperbolic iff diagonalisable in $\mathrm{SL}(2, \mathbb{R})$;
- parabolic iff not diagonalisable in $\mathrm{SL}(2, \mathbb{C})$.

Cayley Hamilton equation: $x^2 - \mathrm{tr}(x)x + 1 = 0$, relates to trace.

Euclidean plane: Group G of isometries, every element fixes a point of plane is there a fixed point (paper by Bruillard and F.)? Answer: Yes. Isometries of plane include rotation, translation, reflection and gluide reflections. However, we may exclude translations and glide reflections. Note we have a positive answer for $\mathbb{R}^2$, $\mathbb{R}^3$, but for $\mathbb{R}^4$ this result does not hold. $x \mapsto Ax + b$, b is a translation and $A \in O(2)$. Then $(x+y)(x+y) = x \cdot x + 2x \cdot y + y \cdot y$, so preserves length since isometry implying it preserves angle.

Fact: Every isometry of $\mathbb{R}^n_{\mathrm{end}}$ is semi-simple.

Exercise: Every isometry of an $\mathbb{R}$-tree is simple.

Proposition 5.13. *If X is a complete* CAT(0)*-space (i.e., a Haadamard space), $\gamma \in \mathrm{Isom}(X)$, then γ is elliptic iff γ has a bounded orbit.*

Proof. If γ has a bounded orbit, then we apply the Bruhat-Tits fixed point theorem. Conversely, if γ is elliptic, then it is fixed. □

We have seen triangles in Euclidean and hyperbolic space. In trees, triangles are tripods. See Figure 40.

If X is a CAT(0)-space, γ hyperbolic isometry. Then, $\min(\gamma)$ is the union of axis of γ ($\gamma \cdot c(t) = c(t+a)$). All axis are parallel (follows from convexity of the distance function; get constant function). Then, $\min(\gamma) \cong Y \times \mathbb{R}$ for some space Y.

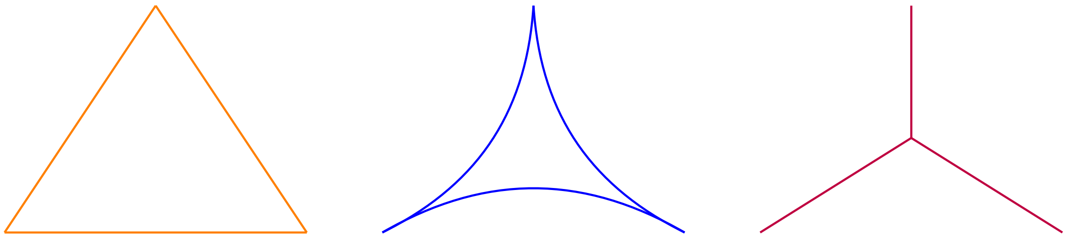


FIGURE 40. Triangles in Euclidean Space, Hyperbolic Space, and Trees

Definition 5.14. A group G is CAT(0), if it admits an action on a CAT(0)-space which is proper, cocompact (i.e., geometric) by isometries.

Examples of CAT(0) groups:

- Coxeter groups (of all types).
- Right angled artin groups.
- Small cancellation groups.
- CAT(-1) groups
  - Finite groups;
  - Finitely generated groups quasi-isometric to free groups;
  - Surface groups, i.e., fundamental groups of surfaces S_g ($g \geq 2$).
- Braid groups, which are related to configuration spaces, in that $B_n = \pi_1(\mathrm{Conf}_n(\mathbb{C}))$. See Figure 41. Relates to resolving the quartic: $(x_1, x_2, x_3, x_4) \mapsto (x_1x_2 + x_3x_4, x_1x_3 + x_2x_4, x_1x_4 + x_2x_3)$ map sending 4 elements to 3 is holomorphic map.

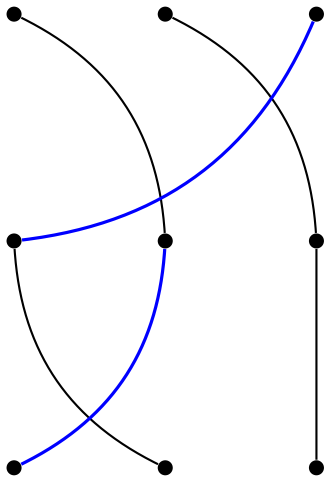


FIGURE 41. Braid group B_3

5.1. **Flat torus theorem (Bridson Haefliger II.7.**

Theorem 5.15 (Flat plane theorem)**.** *A proper cocompact* CAT(0) *space is hyperbolic if, and only if, it does not contain a subspace isometric to* $\mathbb{E}^2$.

Theorem 5.16 (Flat Torus Theorem)**.** *Let A be a free abelian group of rank n acting properly by semi-simple isometries on a complete* CAT(0)*-space X. Then,*

(1) $\min(A) = \bigcap_{\alpha \in A} \min(\alpha)$ *is non-empty and splits as a product* $Y \times \mathbb{E}^n$.
(2) *Every element* $\alpha \in A$ *leaves* $\min(A)$ *invariant (since* A *abelian) and respects the product decomposition.* α *acts on* $Y \times \mathbb{E}^n$ *as the identity on* Y *and as a translation on* $\mathbb{E}^n$.
(3) *The quotient of each* n*-flat* $\{y\} \times \mathbb{E}^n$ *by* A *is an* n*-torus.*

Proof. We prove by induction on n. Recall semi-simple elements are those which are either elliptic or hyperbolic. Cannot be elliptic in this case because acts properly, and elliptic elements fix points. We have infinite order (so must be $\mathbb{Z}^n$). Say $A = \langle \alpha_1, \dots, \alpha_n \rangle$ and $\min(\alpha_1) = Z \times \mathbb{E}^1$ (translation axis). Every $\alpha \in A$ commutes with α_1. Define $N \leq A$ by the elements in A which act trivially on Z. Claim: this group is generated by α_1, i.e., $N = \langle \alpha_1 \rangle$. This claim follows since the action is proper. $A_0 = A/N$ is free abelian of rank $n-1$. Z is a convex subspace, so Z is $CAT(0)$. $\min(A_0) \subset Z$ splits as $Y \times \mathbb{E}^{n-1}$ by induction. This proves (1). □

Lemma 5.17. *If a proper* CAT(0) *space* X *is* cocompact *(i.e.* Isom(X) *the isometry group is cocompact), then there is a bound on the dimension of isometrically embedded flat subspaces* $\mathbb{E}^n \subseteq X$.

Proof. Recall proper means closed balls are compact. If we have $B[x_0, r]$ is compact, can cover by finitely many balls radius $\frac{r}{2}$ (say by N many). Cannot then embed large enough dimension Euclidean spaces. □

Theorem 5.18 (Ascending Chain Condition)**.** *Let* $H_1 \subset H_2 \subset \dots$ *be an ascending chain of virtually abelian subgroups in a group* Γ. *If* Γ *acts geometrically by isometries on a* CAT(0)*-space* X*, then* $H_n = H_{n+1}$ *for large enough* n.

Corollary 5.19. *If a group* Γ *acts geometrically by isometries on a* CAT(0)*-space* X*, then every abelian subgroup of* Γ *is finitely generated.*

Proof. Take $H_n = \langle \alpha_1, \dots, \alpha_n \rangle$ where $\alpha_1, \dots, \alpha_n$ are generators. Must stop at some point, so finitely many generators. □

6. Kleinian Groups

A *Kleinian group* is a discrete subgroup $\mathrm{PSL}(2, \mathbb{C}) = \mathrm{SL}(2, \mathbb{C})/\{\pm I\}$. Why study this?

(1) $\mathrm{PSL}(2, \mathbb{C}) \curvearrowright \widehat{\mathbb{C}}$, where $\widehat{\mathbb{C}}$ is the Riemann Sphere (i.e., the Alexandroff One-Point Compactification of the Complex Plane), and consider stereographic projection.
(2) Hyperbolic 3-manifolds obtained by $\mathbb{H}^3/\Gamma$, where Γ is a Kleinian group (relates to Thurston's Geometrisation Conjecture).

We recall Möbius transformations $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ where $z \mapsto \frac{az+b}{cz+d}$. There are at most two fixed points, since

$$z = \frac{az+b}{cz+d} \implies cz^2 + (d-a)z - b = 0$$

is a quadratic.

Classification of elements:

- Parabolic elements (conjugate to translation map $z \mapsto z+1$), with $\mathrm{Tr}^2(g) = 4$.
- Elliptic elements (corresponds to rotations), with $0 < \mathrm{Tr}^2(g) < 4$.
- Hyperbolic elements (corresponds to dilations), with $\mathrm{Tr}^2(g) > 4$.
- Loxodromic elements (corresponds to a combination of dilation and rotation), with $\mathrm{Tr}^2(g)$ not real.

We have translations, dilations, rotations and inversions. Trace squared conjugacy invariant, in that $\mathrm{Tr}(BAB^{-1}) = \mathrm{Tr}(A)$.

Consider $\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}$, that is, $z \mapsto \frac{az+0}{0z+a^{-1}} = a^2 z$.

A *fractional reflection*, is an orientation reversing map $g : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ defined by $g(z) = \frac{a\overline{z}+b}{c\overline{z}+d}$, where $a, b, c, d \in \mathbb{C}$ such that $ad - bc \neq 0$.

Proposition 6.1. *If g is a fractional reflection, then* $\mathrm{Fix}\,(g)$ *is either empty, one point,* 2 *points, or a circle in* $\widehat{\mathbb{C}}$.

Note. The action is triply transitive.

Proof. Taking $a = 1 = d$ and $b = 0 = c$, we obtain $g(z) = \overline{z}$. If h is any other fractional reflection, we see that hg is a Möbius transformation, and thus has at most two fixed points. Also, can change values of a, b, c, d to obtain examples of each (note g fixes $\mathbb{R}$). Also, can get those of form $g(z) = a\overline{z} + b$. □

Proposition 6.2. *The type (i.e., parabolic, hyperbolic, elliptic, loxodromic) is preserved when taking powers, except if g has finite order for n s.t. $g^n = 1$.*

Proof. By Exercise Sheet 1 Question 1, it follows

$$\mathrm{Tr}(g^2) + \mathrm{Tr}(gg^{-1}) = \mathrm{Tr}^2(g),$$

which implies

$$\mathrm{Tr}^2(g^2) = \left(\mathrm{Tr}^2(g) - 2\right)^2.$$

Comparing the trace and employing induction yields the result. □

Question 6.3. When do two elements commute?

Note. If $fg = gf$ and $g(x) = x$, then $(fg)(x) = f(x) = (gf)(x)$.

Definition 6.4. A *Fuchsian group*, is a discrete subgroup of $\mathrm{PSL}\,(2, \mathbb{R})$.

Definition 6.5. If $G \curvearrowright X$, then G is *freely discontinuous at x*, if there exists a nhood U of x such that $g(U) \cap U = \varnothing$ for each $g \neq 1 \in G$. We denote by Ω°, the set of all freely discontinuous points, called the *free regular set*.

Proposition 6.6 (Page 16, Maskit)**.** Ω°/G *is Hausdorff.*

Proof. Let x and y be inequivalent points (different nhood). We need to find nhoods U of x and V of y such that no translates of U meet V. Notice the translates gU of U are pairwise disjoint. For if $gU \cap hU \neq \varnothing$, then $h^{-1}gU \cap U \neq \varnothing$, implying $h^{-1}g = 1$ (i.e., $g = h$).

Key observation: U has area, and the disjoint translates cover a part of $\widehat{\mathbb{C}}$. It follows the diameter tends to 0, since U has area (and $\widehat{\mathbb{C}}$ has finite surface area). The spherical diameter of a sequence tends to zero. If infinitely many translates of U meets V, then we would have a limit point, which we cannot have. So, only finitely many translates of U meet V, and shrinking appropriately yields the desired result. □

Theorem 6.7 (Jørgensen's inequality (a condition on discreteness))**.** *Let f and g generate a discrete subgroup of* $\mathbb{M}$, *where f is loxodromic, f and g do not share a common fixed point, and g does not keep invariant the fixed point set of f. Then,*

$$\left|\mathrm{Tr}^2(f) - 4\right| + |\mathrm{Tr}([f, g]) - 2| \geq 1.$$

Note. Aside: Riley slice groups generated by two parabolic elements, so conjugate to matrices in $\left\langle \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ \rho & 1 \end{pmatrix} \right\rangle$, where $\rho \in \mathbb{C}$. See Figure 42.

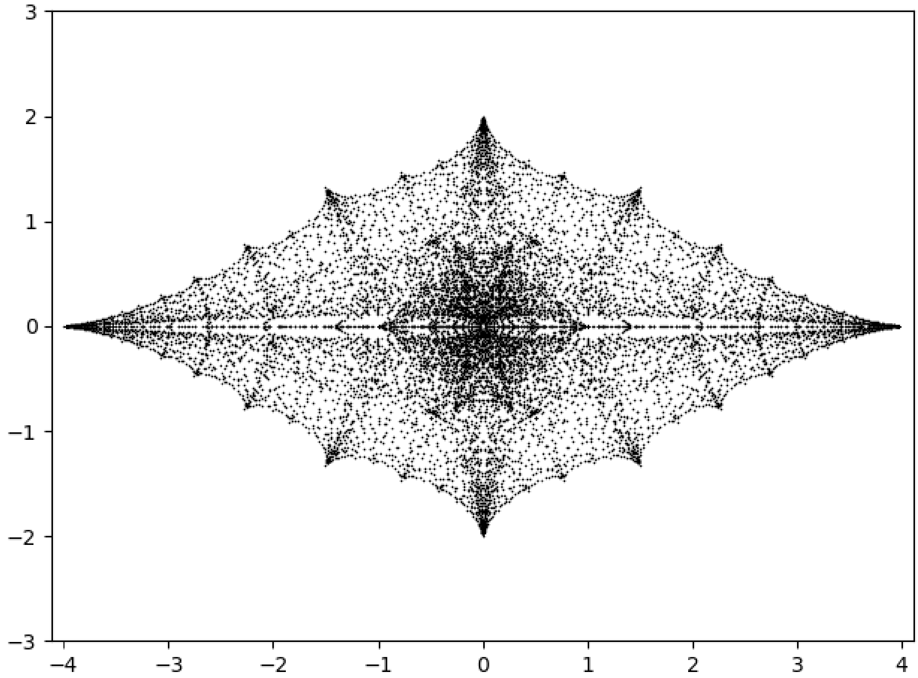


FIGURE 42. The Riley Slice

Definition 6.8. We say x is a *limit point*, if there exists $z \in \Omega^\circ$ with $\langle g_m \mid m \in \mathbb{N}\rangle$ is a sequence of distinct $g_m \in G$ such that $g_m(z) \to x$. We denote by $\Lambda = \Lambda(G)$ the set of limit points, called the *limit set*.

Note. $\Lambda \cap \Omega^\circ = \varnothing$.

Theorem 6.9.
- *Λ is closed, G-invariant and nowhere dense in $\widehat{\mathbb{C}}$.*
- *If Λ contains more than 2 points, then Λ is perfect (i.e., $\Lambda = \Lambda'$, every element in Λ is an accumulation point of other points in Λ).*

Proposition 6.10. *Every perfect set in $\mathbb{R}^n$ is continuum in size.*

Proof. It is known that compact, Hausdorff spaces with no isolated points are at least continuum in size (by embedding the Cantor set into your space). This generalises to locally compact, Hausdorff spaces with no isolated points by considering the Alexandroff One-Point Compactification. Indeed, it is straightforward to check every perfect subspace of $\mathbb{R}^n$ is locally compact, Hausdorff with no isolated points, and thus it easily follows it is continuum in size. □

Note. Terminology: Kleinian groups with $0, 1$ or 2 limit points are called *elementary*. An example of those with 0, are finite groups.

7. Bruhat-Tits Tree

We consider the p-adic numbers $\mathbb{Q}_p$. Bruhat-Tits tree geometric object associated with $\mathrm{SL}\,(2, \mathbb{Q}_p)$. Preview: Ihara's theorem: Discrete torsion-free subgroup of $\mathrm{SL}\,(2, \mathbb{Q}_p)$ is a free group.

The rationals $\mathbb{Q}$ are not complete. We have a completion from $\mathbb{Q}$ to $\mathbb{R}$ (using dedekind cuts or Cauchy sequences). Question: are other completions possible?

Yes: Strowski gives all possible completions. The rationals can be completed to $\mathbb{R}$, or $\mathbb{Q}_p$ for each prime p.

The p-adic-integers: $\mathbb{Z}_p$ are the "allowable sequences" (Serre, A Course in Arithmetic). Fix $p = 5$. Then,

$$\mathbb{Z}/5\mathbb{Z} \longleftarrow \mathbb{Z}/25\mathbb{Z} \longleftarrow \mathbb{Z}/125\mathbb{Z} \longleftarrow \mathbb{Z}/625\mathbb{Z} \longleftarrow \ldots,$$

where $\ldots, 42 \mapsto 17 \mapsto 2$ is a valid sequence. Claim: $\mathbb{Z}_p$ is compact (Tychonoff). $\mathbb{Q}_p$ is a residue field.

Series: $\sum_{k=0}^{\infty} a_k p^k$ diverges in $\mathbb{R}$. We want to ensure this converges. For example, $p = 5$, when we look at 2, 3, and 127, 2 is closer to 127 than 2 is to 3.

p-adic valuation: Take $p = 7$. Then $n = p^k \frac{a}{b}$, where $(a, p) = (b, p) = 1$ (coprime). We have $\nu_p(n) = k$, $\nu_p(x \cdot y) = \nu_p(x) + \nu_p(y)$ and $\nu_p(x + y) \geq \min(\nu_p(x), \nu_p(y))$ (latter can be strict, e.g., $7 = 2 + 5$). Then, $\frac{49}{25}$ goes to 2, $\frac{1}{7}$ goes to -1 and 3 goes to 0. We also have a norm (p-adic norm), where $|x| = p^{-\nu_p(x)}$ (some use base e, instead of p, but nice connections to number theory using p). Then distance is $d(x, y) = |x - y|$.

Discrete valuation on $\mathbb{K}$ is a surjective homomorphism $v : \mathbb{K}^* \to \mathbb{Z}$ satisfying $v(x + y) \geq \min\{v(x), v(y)\}$. $A := \{x \in \mathbb{K} \mid v(x) \geq 0\}$ is a *discrete valuation ring*.

For example, $\mathbb{K} = \mathbb{Q}_p$, $A = \mathbb{Z}_p$.

The *uniformiser* is π such that $v(\pi) = 1$.

For example, if $\mathbb{K} = \mathbb{Q}_p$, then $\pi = p$.

We have $|x| = p^{-v(x)}$ for $x \in \mathbb{K}$, and $d(x, y) = |x - y|$. This is an *ultrametric*: a metric space satisfying $d(x, z) \leq \max\{d(x, y), d(y, z)\}$. In an ultrametric space, triangles are iscoceles, and if two balls intersect, one is contained in the other.

We now define the Bruhat-Tits tree (a Bruhat-Tits building of dimension 1).

- $L = \mathbb{A}e_1 \oplus \mathbb{A}e_2$ (these are lattices)
- Define equivalence relation: two lattices are equivalent in $\mathbb{K}^2$, if $L = \lambda L'$ for some $\lambda \in \mathbb{K}^*$.
- Λ and Λ' are *incident*, if they have representatives L, L' that satisfy $\pi L < L' < L$. Exercise: Show this is symmetric. Suppose Λ and Λ' are incident. Then, $\pi L < L' < L$, implying $\pi\pi^{-1}L' < L < \pi^{-1}L'$, demonstrating it is symmetric.
- Graph: vertices as lattice classes, edges via incidence.

$[e_1, e_2]$ and $[\pi e_1, e_2]$.

Stabiliser of $[e_1, e_2]$ is $\mathrm{SL}(2, \mathbb{Z}_p)$. We have $\mathrm{SL}(2, \mathbb{Q}_p)$, where

$$f_1 := \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} a \\ c \end{pmatrix} \quad \text{and} \quad f_2 := \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} b \\ d \end{pmatrix}.$$

Then, $\mathbb{A} = \mathbb{Z}_p$, where $\mathbb{Z}_p e_1 + \mathbb{Z}_p e_2 \mapsto \mathbb{Z}_p f_1 + \mathbb{Z}_p f_2$ is action of group. Stabalise if they stay in the same class. The tree is $p + 1$ regular.

For $\mathrm{SL}(2, \mathbb{Q}_2)$, we get a $p + 1$ infinite regular tree, as seen in Figure 43. Has no edge inversions. Stabilisers of other vertices; stabilisers are conjugate ($\mathrm{SL}(2, \mathbb{Q}_2)$ preserves parity, so no edge inversions).

Analogue of the upper-half plane. $\mathrm{SL}(2, \mathbb{R})$ acts $\mathbb{H}^2$ and $\mathrm{SL}(2, \mathbb{Q}_p)$ acts on Bruhat-Tits tree ("shadow" of the Drinfold upper half plane). Both examples of Gromov Hyperbolic spaces.

We can extend this to $\widetilde{\mathbb{A}}_n$, where we have $L = \mathbb{A}e_1 \oplus \ldots \oplus \mathbb{A}e_n$. We also have $\widetilde{B}_n$, $\widetilde{C}_n$, $\widetilde{D}_n$, etc, closely related to Lie groups (Dynkin/Coxeter diagrams). Apartments: in $\widetilde{\mathbb{A}}_n$ is equilateral triangle tesselation in the plane. $\widetilde{\mathbb{A}}_2$, $\widetilde{\mathbb{B}}_2$, and $\widetilde{\mathbb{C}}_2$ tile the plane in different ways (triangle, square for first two).

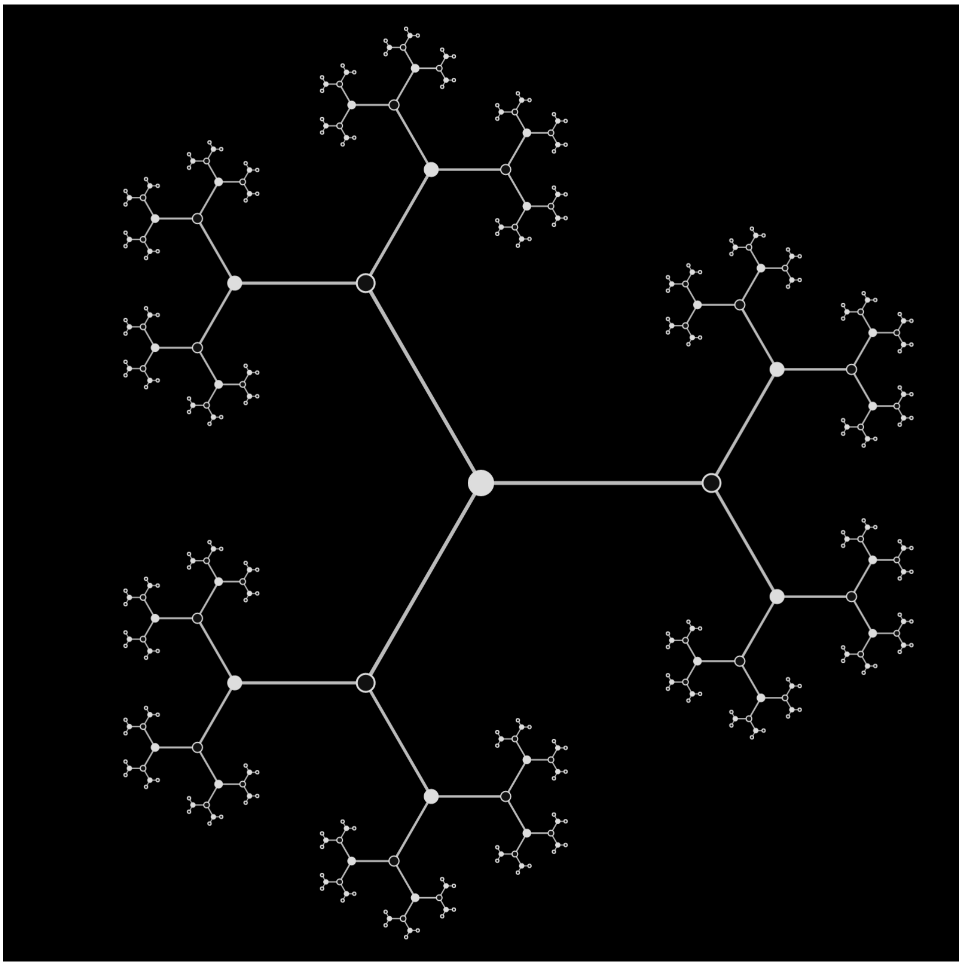


FIGURE 43. Bruhat-Tits Tree $p = 2$

8. Deciding Freeness

Deciding Freeness: Deciding if a given group is free.

Theorem 8.1 (Tits Alternative)**.** *Every finitely generated metric group is either virtually solvable or contains a free non-abelian subgroup.*

8.1. **History.** Let

$$F_{\alpha,\beta} := \left\langle A = \begin{bmatrix} 1 & \alpha \\ 0 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 \\ \beta & 1 \end{bmatrix} \right\rangle,$$

where $\alpha, \beta \in \mathbb{R}$.

Question 8.2. When is $F_{\alpha,\beta}$ free?

Sanov (1947): Yes for $\alpha = \beta = 2$ (by Ping-Pong, as seen before).
Brenner (1955): Yes for $\alpha = \beta \geq 2$.
Chang, Jennias and Ree (1958): Yes for $|\alpha\beta| \geq 2$, $|\alpha\beta - 2| \geq 2$ and another condition. No for infinite families of $F_{\alpha,1}$. Note $F_{\alpha,\beta} \cong F_{\alpha\beta,1}$.
Open problem: Is it free for all values in $\alpha = \beta \in (-2, 2) \cap \mathbb{Q}$?
Newman (1967): $\left\langle \begin{bmatrix} -a & b \\ -c & d \end{bmatrix}, \begin{bmatrix} -\alpha & -\beta \\ \gamma & \delta \end{bmatrix} \right\rangle$ is free for $a, b, c, d, \alpha, \beta, \gamma, \delta \geq 0$, $d - \alpha \geq 2$, $\delta - \alpha \geq 2$.

Lyndon, Ullman (1967): Rederives Brenner, Newman, etc. results using Theorem of Macbeth using geometric results (Klein-Maskit Combination Theorems).

Purzizsky, Rosenburger, Kern-Ioterner (1972-1976): Classification of 2-generated discrete subgroups of $\mathrm{PSL}\,(2,\mathbb{R})$. Trace-minimising algorithm: (A, B), $\mathrm{tr}(A) \leq \mathrm{tr}(B) \leq \mathrm{AB}, \mathrm{tr}\big(AB^{-1}\big), \ldots$, such that the trace grows as the length of words grow. Acting on hyperbolic plane, by finding the "fundamental domain". Hyperbolic action: translation axis. Let $g \in \mathrm{PSL}\,(2,\mathbb{R})$; hyperbolic if g fixed 2 points on $\partial\mathbb{H}^2$.

Gilman, Maskit (1989): Wrote explicit algorithm to decide whether $\langle A, B\rangle \subset \mathrm{PSL}\,(2,\mathbb{R})$ generate a discrete subgroup.

Eich, Kirschner, Leadman (2014) and Kirschner, Ruther (2017): Same thing, plus constructive membership problem for 2-generated discrete subgroups of $\mathrm{PSL}\,(2,\mathbb{R})$.

Constructive membership problem: $X \subset \mathrm{PSL}\,(2,\mathbb{R})$, $G = \langle X\rangle$. Given $g \in \mathrm{PSL}\,(2,\mathbb{R})$, is $g \in G$ and if so, what is g as a word in X?

Algorithm to decide whether finite $X \subset \mathrm{PSL}\,(2,\mathbb{R})$ generates a discrete group (Riley, 1983 - refined in 1984).

Fundamental domain: G discrete acts on $\mathbb{H}^2$. Make balls large as possible, disjoint from others; will be tiling of $\mathbb{H}^2$ which is a fundamental domain.

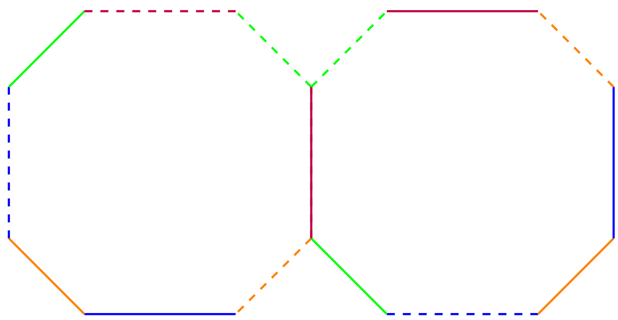


FIGURE 44. Side pairings

Theorem 8.3 (Poincare Polygon Theorem)**.** *Polygon P in $\mathbb{H}^2$ (works in higher dimensions). Side pairings, and compatibility conditions as in Figure 44 (angles add to 2π for proper tiling). Then, P is a fundamental domain for some $G \leq \mathrm{PSL}\,(2,\mathbb{R})$. G is generated by isometries match edges; cycles iff relation in G.*

Ford domain: generators are isometric circles. Exterior of all isometric circles in group. Fundamental domain is excluded region, as in Figure 45.

Dirchlet domain: Fix $v \in \mathbb{H}^2$. Generator g: perpendicular bisector of $[v, gv]$, as in Figure 46.

Recall upper half plane model: $\{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$. Isometries correspond to Moebius transformation: $g \mapsto \frac{az+b}{cz+d}$, where $a, b, c, d \geq 0$ and $ad - bc = 1$. Poincare disk model: unit disk. Both conformal (preserve angles), and geodesics are circles boundary at right angle.

Theorem 8.4. *Finitely generated Fuchsian groups have finite sided Ford/Dirchlet domains.*

Algorithm to decide whether finite $X \subset \mathrm{PSL}\,(2,\mathbb{R})$ (undecidable if $\mathbb{R}$ replaced by $\mathbb{C}$) generate a discrete group (Riley, 1983). Downside: Exponential search.

$X \subset \mathrm{PSL}\,(2,\mathbb{R})$, $G = \langle X\rangle$. T is the set of all words in X of length $\leq n$. Prune T to obtain sides of a polygon $T \to T'$. Test if corresponding polygon is a fundamental domain (rather, a Poincaré polygon), and $\langle T'\rangle = \langle X\rangle$ (constructive

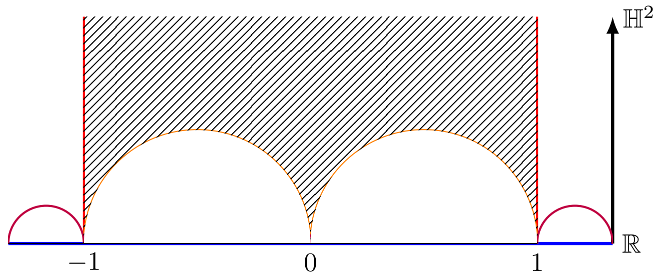


FIGURE 45. Ford domain

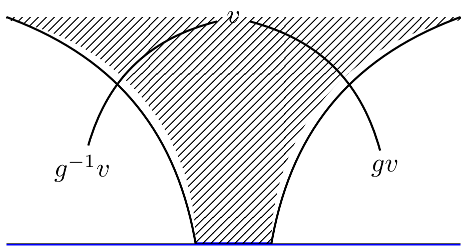


FIGURE 46. Dirchlet domain

membership test). $g \in X$, is $g \in \langle T' \rangle$? Series of tiles from v to gv. Multiply by generators of edges in tiles.

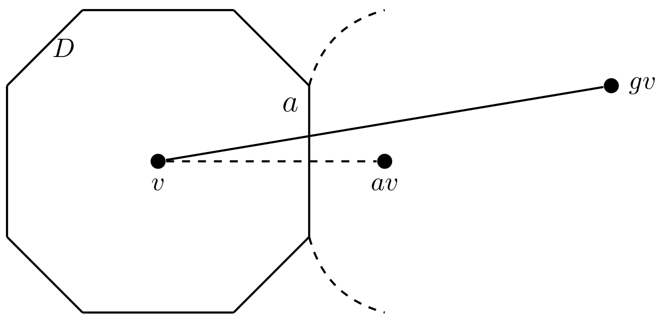


FIGURE 47. Constructive membership test

See Figure 47;

$$d(gv, av) < d(gv, v) \longrightarrow d\big(a^{-1}gv, v\big) < d\big(a^{-1}gv, a^{-1}v\big) = d(gv, v).$$

Take $g \leftarrow a^{-1}gv$. Then, $w = a_1 a_2 \dots a_k$ where $w^{-1}gv \in D$. We have $g \in G$ iff $w = g$. Now, if yes, we are done. If no, increase n and try again.

Jiang (2001): GM-Algorithm has poly. complexity.

What if G is indiscrete? We recall Jorgensen's Inequality. We also have the following.

Lemma 8.5 (Shimizu's Lemma)**.** *If* $X = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, $Y = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, *then* $|c| \geq 1$.

κ non-archimidean local field (e.g. $\mathbb{Q}_p$). O_κ valuation ring. $T = (V, E)$. The vertices V are the lattices Λ of rank 2 up to scalar multiplication over valuation ring. The edges E are (Λ, Λ') are $\pi L < L' < L$.

$\begin{bmatrix} a & c \\ b & d \end{bmatrix} \sim \begin{bmatrix} 1 & 0 \\ u & p^n \end{bmatrix}$ where $n \in \mathbb{Z}$ and $u \in \mathbb{Q}$. Boundary of Bruhat-Tits tree is $\mathbb{Q}_p$ with infinity. For $u \mod p^n \mapsto \frac{au+b}{cu+d}$.

Matthew Conder (2022): Algorithm to decide discreteness and freeness of 2- and 3-generated subgroups of $\mathrm{Aut}(T)$, T locally finite tree. Conjectured an algorithm to every finitely generated subgroup of $\mathrm{Aut}(T)$.

Proposition 8.6. *Discrete and free implies no elliptic elements.*

Ari M (2023): Alternative algorithm to decide simultaneous discreteness and freeness of finitely generated subgroups of $\mathrm{Aut}(T)$.

$F_n = \langle a_1, \ldots, a_n \rangle$. $X = \langle w_1, \ldots, w_m \rangle \subset F_n$. $G = \langle X \rangle$. $|\cdot|$: word length. $X \to X'$ if $|w_i w_j| < |w_i|$: replace w_i with $w_i w_j$.

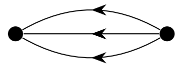
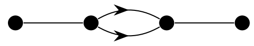
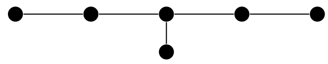
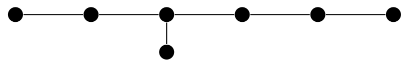
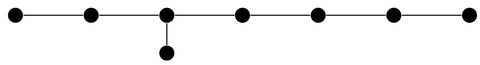
9. Tits Buildings

In the 19th century Klein's Erlanger Program: study geometry by means of their symmetries (1872), later generalised by Elie Cartan. Also, the study of Lie Groups by Sophus Lie.

- Lie group: group which is also a differentiable manifold with continuous inversion and multiplication operations.
- They form natural models for continuous symmetry, e.g. rotational symmetries in 3 dimensions are described by SO(3).
- Lie algebra: linear object that can be canonically attached to a Lie group and contains a lot of information about it.
- Semisimple Lie algebras over an algebraically closed field of characteristic zero are completely classified by their root system (Killing-Cartan), which are in turn classified by Dynkin diagrams.

Example 9.1. $X_0 x_1 + X_2 X_3 = 0$ over $\mathbb{C}$. Then, $(X_0 + iX_1)(X_0 - iX_1) + X_2 X_3 = 0$. So, $X_0^2 + X_1^2 + X_2 X_3 = 0$. Can do same for X_2 and X_3; these are forms.

Theorem 9.2. *Classification of connected Dynkin diagrams (equivalently, irreducible root systems, and also simple complex Lie algebras):*

Dynkin diagram	Notation	Simple Lie algebra
	$A_n, n \geq 1$	$\mathfrak{sl}_{n+1}(\mathbb{C})$
	$B_n, n \geq 2$	$\mathfrak{so}_{2n+1}(\mathbb{C})$
	$C_n, n \geq 2$	$\mathfrak{sp}_{2n}(\mathbb{C})$
	$D_n, n \geq 3$	$\mathfrak{so}_{2n}(\mathbb{C})$
	G_2	$\mathfrak{g}_2$
	F_4	$\mathfrak{f}_4$
	E_6	$\mathfrak{e}_6$
	E_7	$\mathfrak{e}_7$
	E_8	$\mathfrak{e}_8$

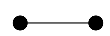
Theorem 9.3. *Let κ be an algebraically closed field of characteristic zero. Then every almost simple linear algebraic group is isogenous to exactly one of the Dynkin diagrams.*

Buildings: A geometric theory of algebraic groups. Jacques Tits (Abel prize 2008, Wolf Prize 1993) gave a converse to Klein's Erlanger programm: study groups by means of their geometries.

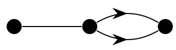
Coxeter groups as abstraction of Weyl groups.

- *Coxeter Group*: $W = \left\langle S \;\middle|\; (s_i s_j)^{m(s_i,s_j)} = 1 \right\rangle$, where $m(s_i, s_i) = 1$ (so all of the generators are involutions), $m(s_i, s_j) = m(s_j, s_i)$ and $2 \leq m(s_i, s_j) \leq \infty$ for $i \neq j$. We always assume S is finite.
- *Coxeter Diagram*: Draw one node for each generator s_i and then join s_i to s_j (labeled) iff $m(s_i, s_j) \geq 3$. These are intimately linked with the Dynkin diagram.

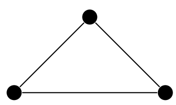
Example 9.4. $W = \left\langle s, t \;\middle|\; s^2 = t^2 = (st)^3 = 1 \right\rangle$ has the following Coxeter diagram.



Example 9.5. $W = \left\langle s, t, u \;\middle|\; s^2 = t^2 = u^2 = (st)^3 = (tu)^4 = (su)^2 = 1 \right\rangle$ has the following Coxeter diagram.



Example 9.6. $W = \left\langle s, t, u \;\middle|\; s^2 = t^2 = u^2 = (st)^3 = (tu)^3 = (su)^3 = 1 \right\rangle$ has the following Coxeter diagram.



The *Coxeter complex* of a Coxeter system (W, S):

- $W_J = \langle J \rangle$, where $J \subseteq S$, is a *standard subgroup*.
- $\Sigma(W, S)$: poset of standard cosets in W, ordered by reverse inclusion. Thus, $B \leq A$ in Σ if, and only if, $A \subseteq B$ as subsets of W, and we call B a *face* of A.
- $\Sigma(W, S)$ is called the *Coxeter complex* associated to (W, S).

Example 9.7. $W = \left\langle s, t \;\middle|\; s^2 = t^2 = (st)^3 = 1 \right\rangle$. The standard subgroups are 1, $\{1, s\}$, $\{1, t\}$ and W, and for example the faces of $\{t\}$ are $\{1, t\}$ and $\{t, ts\} = t\{1, s\}$.

Definition 9.8. Let V be a real vector space with basis $\{(e_i)_{i=1}^n\}$. Define a symmetric bilinear form on V by $B(e_i, e_j) = -\cos\left(\frac{\pi}{m_{ij}}\right)$. The *geometric representation of W* on V is defined by $s(v) = v - 2B(v, e_s)e_s$.

Note. No information is lost (i.e., representation is faithful Tits). B is positive definite if, and only if, W is finite; in this case we say W is *spherical*. If B is positive semi-definite of corank 1, we say W is *Euclidean*. If B is indefinite, we say W is *hyperbolic*. Coxeter groups are linear over a field of characteristic zero and by our finite generation assumption and thus virtually torsion-free (Selberg 1960) and residually finite (Malcev 1940).

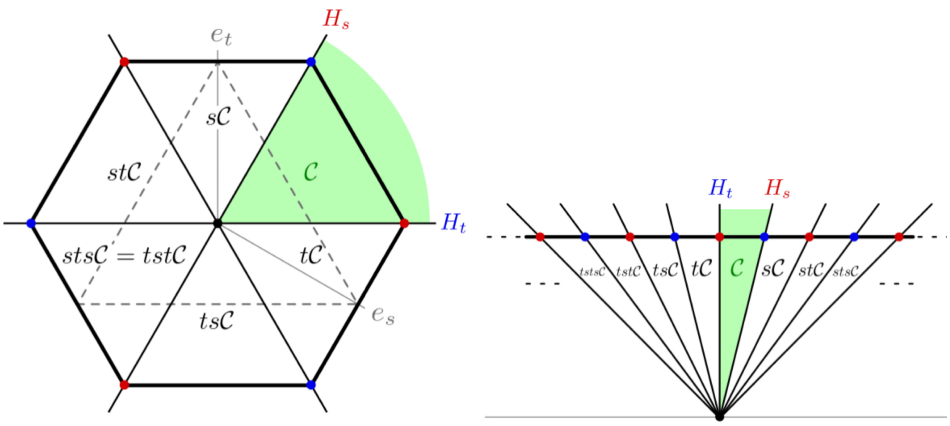


Figure 48. Coxeter Complexes

Figure 48 shows two classic examples of Coxeter complexes. The diagram on the right corresponds to the infinite dihedral group $D_\infty = \langle s, t \mid s^2 = t^2 = 1 \rangle$. Taking the dual of the diagram on the left yields Figure 49. The Cayley graph is dual of Coxeter complex.

Figure 50 has $EFE = FEF$, $VEVE = EVEV$ and $VF = FV$. It is a Spherical Coxeter complex, with the third cube the following Dynkin diagram.

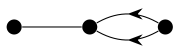


Figure 51 shows another Euclidean Coxeter complex.

- W: group of isometries of the plane generated by the (affine) reflections with respect to the sides of an equilateral triangle.
- Example of a Euclidean reflection group.
- $W := \left\langle s, t, u \;\middle|\; s^2 = t^2 = u^2 = (st)^3 = (tu)^3 = (su)^3 = 1 \right\rangle$; recall the Coxeter diagram is K_3.

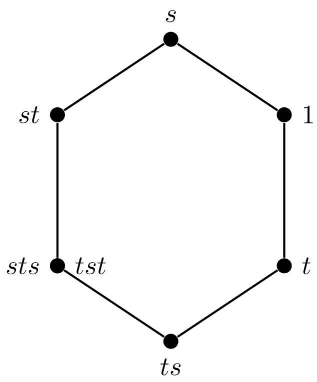


FIGURE 49. Dual is Cayley graph

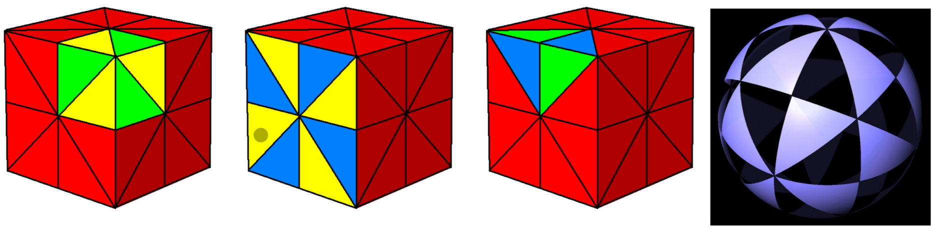


FIGURE 50. Spherical example coming from the cube

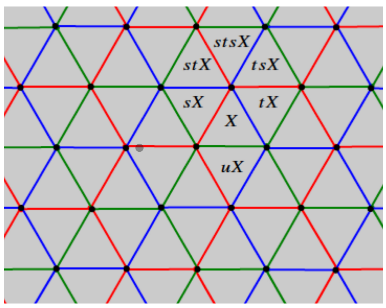


FIGURE 51. Euclidean Coxeter complex

- Coxeter complex: plane tiled by equilateral triangles.

Definition 9.9. A *building* is a simplicial complex Δ that can be expressed as the union of sub complexes Σ (called *apartments*) satisfying the following axioms:

($B0$) Each apartment Σ is a Coxeter complex.
($B1$) For any two simplicies $A, B \in \Delta$, there is an apartment Σ containing them.
($B2$) If Σ and Σ' are two apartments containing A and B, then there is an isomorphism $\Sigma \to \Sigma'$ fixing A and B pointwise.

A building Δ associated to a vector space V:

- V: $n \geq 2$-dim vector space over an arbitrary field k.
- $\mathbb{P}(V)$: non-zero subspaces of V (recall projective space is where we remove the identity and points are identified up to scalar multiplication - the result is where points are now lines through the origin).

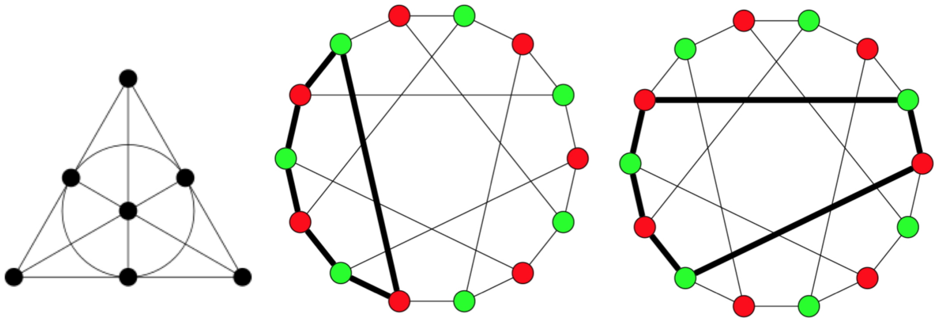


FIGURE 52. Fano Plane

- $\Delta = \Delta(V)$: flag complex of $\mathbb{P}(V)$; thus the simplices are chains $V_1 < V_2 < \ldots < V_k$ of non-zero proper subspaces of V. The maximal simplices, called *chambers*, are the chains $V_1 < \ldots < V_{n-1}$ such that $\dim V_i = i$.

Assume Δ is a simplicial building of type (W, S) with a type preserving action of G on it. Suppose $\mathcal{A}$ is a G-invariant system of apartments. We say the G-action is *strongly transitive* (with respect to $\mathcal{A}$) if G acts transitively on the set of pairs (Σ, C) consisting of an apartment $\Sigma \in \mathcal{A}$ and a chamber $C \in \Sigma$.

Assume the G-action is strongly transitive, and choose an arbitrary pair (Σ, C) as in the definition, we will refer to Σ as the *fundamental apartment* and to C as the *fundamental chamber*.

Consider $\mathbb{R}^3$, which has e_1, e_2, e_3 as standard basis. Then $\langle e_1 \rangle \subset \langle e_1, e_2 \rangle \subset \ldots$ yields a flag complex. We have $\mathrm{GL}\,(n+1, k) \to \mathbb{P}^n(k)$.

The triangle with vertices e_1, e_2 and e_3 yield a hexagon (the 3×3 matrices which preserve e_1, e_2, e_3 are upper triangular matrices).

Algebraic groups correspond to groups with a "BN Pair", which correspond to spherical buildings.

The action of special subgroups B, N and T.

- Σ: that of standard basis, C: edge joining $[e_1]$ to $[e_1, e_2]$.
- $B := \{g \in G \mid gC = C\}$: upper triangular matrices.
- $N := \{g \in G \mid g\Sigma = \Sigma\}$: monomial matrices.
- $T := \{g \in G \mid g$ fixes Σ pointwise$\}$: diagonal matrices.
- $W = N/T = \left\langle s, t \;\middle|\; s^2 = t^2 = (st)^3 = 1 \right\rangle$ where $s = (12)$, $t = (23)$.

A group G has a BN pair of subgroups B and N if the following hold:

- $G = \langle B, N \rangle$
- $T := B \cap N \leq N$
- $W := N/T$ with set generators S
- For $s \in S$ and $w \in W$ one has $sBw \subset BswB \cup BwB$
- For $s \in S$ one has $sBs^{-1} \not\leq B$

W is called the *Weyl group* and (G, B, N, S) a *Tits system*. Side note: Bruhat decomposition: $G = \amalg_{w \in W} BwB$.

Theorem 9.10. *Given a BN-pair in G, the generating set S is uniquely determined, and (W, S) is a Coxeter system. There is a thick building $\Delta = \Delta(G, B)$ that admits a strongly transitive G-action such that B is the stabiliser of a fundamental chamber and N stabilises a fundamental apartment and is transitive on its chambers.*

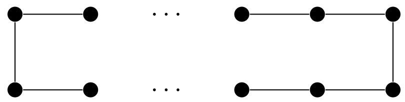
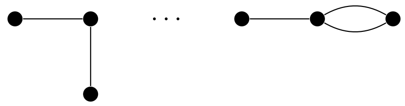
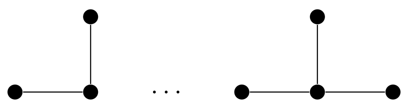
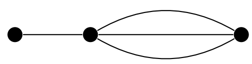
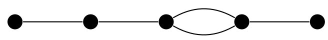
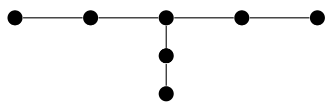
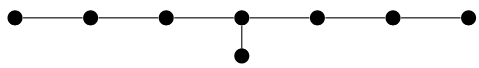
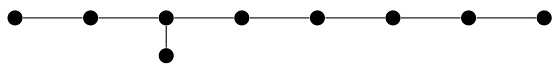
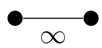
Theorem 9.11. *Suppose a group G acts strongly on a thick building Δ with fundamental apartment Σ and fundamental chamber C. Let B be the stabiliser of C,* *and let N be the subgroup of G that stabilises Σ and is transitive on the chambers of Σ. Then, (B, N) is a BN-pair in G and Δ is canonically isomorphic to $\Delta(G, B)$.*

Theorem 9.12 (Tits '74)**.** *Thick, irreducible, spherical buildings of rank at least 3 are either*

- *Classical buildings (associated to classical groups); or*
- *Algebraic buildings (associated to algebraic groups); or*
- *Mixed buildings (associated to mixed groups).*

Restriction to rank at least three is needed, as there are free constructions in rank two. Moreover, classifying finite buildings of type A_2 is equivalent to classifuing finite projective planes, a well-known problem which is out of reach.

The following is a classification of Euclidean buildings.

Diagram	*Notation*
	$\tilde{A}_p, p \geq 2$
	$\tilde{B}_p, p \geq 3$
	$\tilde{C}_p, p \geq 2$
	$\tilde{D}_p, p \geq 4$
	$\tilde{G}_2$
	$\tilde{F}_4$
	$\tilde{E}_6$
	$\tilde{E}_7$
	$\tilde{E}_8$
	$\tilde{A}_1$

- Euclidean building of dimension at least three is a Bruhat-Tits building (Tits '86).
- Building at infinity of Bruhat-Tits building is Moufang.
- Tits-Weiss: Classification of Moufang polygons.
- Artin-Zorn: Every finite alternative division ring is a field.

Bruhat-Tits buildings:

- Introduced to study of reductive algebraic groups over valued fields with not necessarily discrete valuation.

- Important subclass when valuation is discrete (seen via geometric realization): simplicial Euclidean buildings (only ones known before Bruhat-Tits '72).
- Let $\mathbb{L}$ be a locally compact, non-discrete topological field. Then $\mathbb{L}$ is $\mathbb{R}$, $\mathbb{C}$, or a finite extension of either $\mathbb{Q}_p$ or $K = \mathbb{F}_p((t))$.

Theorem 9.13 (Martin, JS, Steinke, Struyve)**.** *The Bruhat-Tits building is metrically complete if, and only if, the associated (skew) field is spherically complete, up to certain cases involving infinite dimensionality and residue characteristic two.*

Recall a discrete valuation on $\mathbb{K}(\mathbb{Q})$ is a surjective homomorphism $\nu : \mathbb{K}^* \to \mathbb{Z}$, satisfying

$$\nu(x+y) \geq \min(\nu(x), \nu(y)).$$

$x \in \mathbb{Q}^\star$ written uniquely as $x = p^n u$, p-adic valuation: $\nu(x) = n$.

- $A := \{x \in \mathbb{K} \mid \nu(x) \geq 0\}$ is a *discrete valuation ring*. Fractions a/b with b coprime to p.
- $\mathbb{K}$ is the field of fractions of A.
- Uniformiser: π such that $\nu(\pi) = 1$ for $\mathbb{Q} : \pi = p$.
- Residue field: $k = A \setminus \pi A$ for $\mathbb{Q}$: $k = \mathbb{F}_p$.

Discrete valuations yield a second BN pair for $\mathrm{SL}(n, \mathbb{K})$:

- First observed by Matsumoto and Iwahori then vastly generalised by Bruhat and Tits.
- B: inverse image in $\mathrm{SL}(n, A)$ of upper triangular matrices in $\mathrm{SL}(n, k)$.
- N: monomial subgroup of $\mathrm{SL}(n, \mathbb{K})$.
- $T = B \cap N$ is diagonal subgroup of $\mathrm{SL}(n, A)$, conjugation action of N on T permutes the diagonal entries.
- Short exact sequence:
$$1 \to T(\mathbb{K})/T(A) \to W := N(\mathbb{K})/T(A) \to N(\mathbb{K})/T(\mathbb{K}) \to 1.$$
- $W \cong (\mathbb{K}^\star/A^\star)^{n-1} \rtimes S_n$.

Bruhat-Tits tree (Bruhat-Tits building of dimension 1):

- LatticeL $L = Ae_1 \oplus Ae_2$.
- Call two A-lattices *equivalent* in $\mathbb{K}^2$, if $L = \lambda L'$ for some $\lambda \in \mathbb{K}^*$.
- Type of $[[f_1, f_2]]$ as $v(\det(f_1, f_2)) \mod 2$.
- Distinct lattice classes Λ, Λ' are *incident*, if they have representatives that satisfy $\pi L < L' < L$.
- This relationship is symmetric because $\pi L' < \pi L < L'$.
- Graph: Vertices are lattice classes, edges via incidence. This is a tree, called the *Bruhat-Tits tree*.

$\mathrm{SL}(2, \mathbb{Q}_p)$ has local building $\mathrm{SL}(2, \mathbb{F}_p)$ and global building $\mathrm{SL}(2, \mathbb{Q}_p)$.
Relating the Bruhat-Tits tree to the second BN pair:

- C: edge given by $[e_1, e_2]$ and $[e_1, \pi e_2]$
- Stabiliser of $[e_1, e_2] = \mathrm{SL}(2, A)$, stabiliser of $[e_1, \pi e_2]$ is $g\mathrm{SL}(2, A)g^{-1}$ where $g = \begin{pmatrix} 1 & 0 \\ 0 & \pi \end{pmatrix}$, their intersection is B.
- Fundamental apartment obtained by applying N to C, we get $[[\pi^a e_1, \pi^b e_2]]$, $a, b \in \mathbb{Z}$. Arbitrary apartment $g\Sigma$ is similar, but with e_1, e_2 replaced by an arbitrary basis of $\mathbb{K}^2$.
- The Bruhat-Tits tree was crucial to the construction of Ramanujan graphs by Lubotzky-Philips-Sarnak and Bruhat-Tits buildings are used to construct high-dimensional expanders.

Theorem 9.14 (Ihara (Serre))**.** *Every discrete torsion-free subgroup* Γ *of* $\mathrm{SL}(2,\mathbb{Q}_p)$ *is free.*

Proof. A group acting freely on a tree (no inversions, trivial point stabilisers) is a free group (Basse-Serre). Note that the action of $\mathrm{SL}(2,\mathbb{Q}_p)$ on the Bruhat-Tits tree is type-preserving so there are no edge inversions. Assume $H \leq \Gamma$ fixes a vertex of the Bruhat-Tits tree. H is bounded and hence relatively compact, thus compact. H is compact and discrete, hence finite, and thus trivial. Note $\mathrm{SL}(2,\mathbb{Z}_p)$ is compact since $\mathbb{Z}_p$ is compact, so the point stabiliser is compact. □

Euclidean Buildings are examples of CAT(0)-spaces.

- Given x, y, z in X, the triangle inequality implies there is a *comparison triangle* in the Euclidean plane $\mathbb{R}^2$ (unique up to an isometry of $\mathbb{R}^2$).
- Given a geodesic $[x, y]$ and a point $p = p_t \in [x, y]$, there is a corresponding point $\overline{p} = \overline{p}_t$ on the line segment $[\overline{x}, \overline{y}]$ in $\mathbb{R}^2$.
- A metric space is CAT(0) is for any $x, y \in X$ there is a geodesic $[x, y]$ such that: For all $p \in [x, y]$ and all $z \in X$ one has $d_X(z, p) \leq d_{\mathbb{R}^2}(\overline{z}, \overline{p})$.
- Examples include: Euclidean spaces, Hilbert spaces, Riemannian symmetric spaces of non-positive curvature, Euclidean buildings.
- Let X be a locally compact CAT(0) space of geometric dimension n. If any two points are contained in a common n-flat, then X is the metric realisation of a Euclidean building (Kleiner).

Elie Cartan: If G is a compact group of isometries of a complete simply connected Riemannian manifold M of non-positive curvature, then G fixes a point of M.

Theorem 9.15 (Bruhat-Tits fixed point theorem)**.** *Let* G *be a group of isometries of a complete* $CAT(0)$ *space* X*. If* G *stabilises a non-empty bounded subset of* X*, then* G *fixes a point of* X*.*

Application: Every compact subgroup of $\mathrm{SL}(n,\mathbb{R})$ is conjugate to a subgroup of $\mathrm{SO}_n(\mathbb{R})$ using symmetric space and can obtain a p-adic analogue from the Bruhat-Tits building.

Theorem 9.16 (Serre)**.** *If* X *is a complete* CAT(0) *space, then every non-empty bounded subset of* A *admits one and only one circumcenter.*

BN pair for an algebraic group:

- $G(k)$: k-rational points of a (connected) reductive linear algebraic group G, T: maximal k-split torus, N: normaliser in G of G.
- Grothendieck: Any smooth connected affine group G over a field k contains a k-torus T such that $T_{\overline{k}}$ is maximal in $G_{\overline{k}}$.
- B: Borel subgroup B in G, i.e., B is minimal such that G/B is a projective variety.
- Borel-Tits: $(B(k), N(k))$ is a BN-pair for $G(k)$ relying on the crucial result by Grothendieck.
- Tits: uniform proof of the simplicity (modulo center) of the groups of rational points of irreducible isotropic simple groups (over sufficiently large fields).

Spherical buildings from algebraic groups: Let $\Delta = \Delta(G)$ be the simplicial complex whose simplexes correspond to proper k-parabolic subgroups of G as follws:

- The vertices of Δ correspond to maximal (proper) k-parabolic subgroups of G and chambers to minimal parabolic subgroups.
- Vertices $Q_1, \ldots, Q_m$ for the vertices of a simplex iff $\bigcap_{i=1}^m Q_i$ is a k-parabolic subgroup, which corresponds to the simolex σ.

- For any maximal k-split torus T of G, there are only finitely many k-parabolic subgroups containing T, and their corresponding simplices in Δ form a Coxeter complex (an apartment) whose Coxeter group is $W = N(T)/T$.
- $G(k)$ acts on the set of k-parabolic subgroups by conjugation and hence acts on the building $\Delta(G)$ by simplicial automorphisms.

(W, S) any Coxeter system (with S finite), and X a connected Hausdorff space. The *mirror structure on X over S* is a collection $(X_s)_{s\in S}$ where each X_s is closed and non-empty, for which we call X_s the s-mirror of X.

For each $x \in X$ define $S(x) \subset S$ by $S(x) = \{s \in S \mid x \in X_s\}$. Define $\sim$ on $W \times X$ by $(w, x) \sim (w', x')$ if, and only if, $x = x'$ and $w^{-1}w' \in W_{s(x)}$. Then define $\mathcal{U}(W, X) = W \times X/\sim$ equipped with the quotient topology.

(1) Cayley graphs: obtained from the "star".
(2) Coxeter complexes: obtained from the "triangle".
(3) $\mathcal{U}(W, X)$ is connected, Hausdorff and with X as the fundamental domain for the natural action of W, i.e., $\mathcal{U}(W, X)/W = X$.

The *nerve* $L(W, S)$ of (W, S) is the simplicial complex with simplex σ_T for each $T \subseteq S$ such that T is non-empty and W_T is finite.

The *chamber* K is the cone of the barycentric subdivision L' of the nerve $L = L(W, S)$. For each $s \in S$, define $K_s \subseteq K$ to be the closed star in L' of the vertex s.

The Davis complex:

- Connected, Hausdorff, locally finite.
- W-action on Σ is properly discontinuous with quotient K, and all point stabilisers are conjugates of finite special subgroups of W.
- Contractible so in particular simply connected.
- CAT(0) using the Cartan-Hadamard theorem and the Gromov link conidition.
- If a group G acts geometrically by isometries on a CAT(0) space, then the word problem and conjugacy problem are both solvable for G.

Theorem 9.17 (Cartan-Hadamard)**.** *Let X be a complete, connected geodesic metric space. If X is locally* CAT(0)*, then the universal cover of X is* CAT(0)*.*

Theorem 9.18 (Gromov Link Condition)**.** *If X is piecewise Euclidean polyhedral complex, then X is locally* CAT(0) *if, and only if, every vertex v of X, the link of v in X is* CAT(1)*.*

Note. For example, a vertex of a cube is incident to three faces each at an angle of $\frac{\pi}{2}$, yielding a sum of $\frac{3\pi}{2} > \pi$.

10. Algorithmic problems in combinatorial group theory

Ari M: PSL $(2, k)$ (where k is $\mathbb{R}$ or non-archimidean field).

Throughout, let $G = \langle S \mid R\rangle$ be a finitely presented/generated group. For each problem, we are asked to construct a Turing machine or prove its non-existence.

Word problem: Determine if a given word w in X represents the trivial element of G, i.e., if $w \in \langle\langle R\rangle\rangle$ (normal closure of R).

Conjugacy problem: Given v, w in X, determine if there exists $g \in G$ such that $[w] = g^{-1}[v]g$. Note while this is difficult in general, there are instances where it can be easy: for a concrete example, consider matrices, as conjugate matrices have the same trace. An extension of this problem is that of the simultaneous conjugacy problem.

Trviality problem: Is G trivial or not?

Embedding problem: Is G embedded into H?

Membership problem: $H \leq G$. If $g \in G$, is $g \in H$?

Constructive membership problem: $H \leq G$. If $g \in G$, is $g \in H$, and if so, what is g as a word in H?

Theorem 10.1 (Novikov, Boone 1950's)**.** *All of the above are algorithmically undecidable.*

Free groups of finite rank have solvable word problem (reduce word, if identity, yes; otherwise no).

11. Hyperbolic Groups

11.1. The boundary at infinity of a CAT(0)-space.

Definition 11.1. Given two rays $c, c' : [0, \infty) \to X$, we say c and c' are *asymptotic*, if there exists $K \geq 0$ such that $d(c(t), c'(t)) \leq K$ for each $t \in [0, \infty)$.

Note. This defines an equivalence relation. Moreover, as an example, in the Euclidean plane the asymptotic rays are precisely those which are parallel.

We denote by ∂X the *boundary* of X, which is the set of equivalence classes of rays. Let $\overline{X} := X \cup \partial X$.

Lemma 11.2. *Given $\epsilon > 0$, $a > 0$, $s > 0$ there exists $T = T(\epsilon, a, s) > 0$ such that if x, x' are points with $d(x, x') = a$, $d(\sigma_t(s), \sigma_{t+t'}(s)) < \epsilon$ for each $t \geq T$ and $t' > 0$.*

Note. σ_t is the geodesic from x' to $c(t)$.

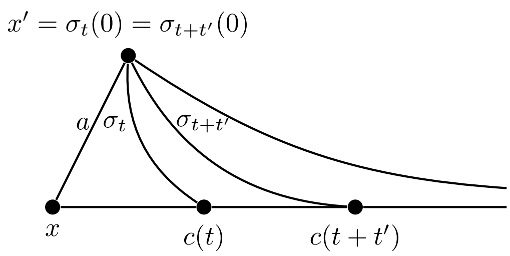


Figure 53. Unique asymptotic ray

Proposition 11.3. *If X is a complete* CAT(0) *space, and $c : [0, \infty) \to X$ is a geodesic ray "issuing at x" (i.e., starting at x), and $x' \in X$, then there is a unique geodesic ray issuing from x' and asymptotic to c.*

Proof. Uniqueness follows from the convexity of the distance function. For uniqueness, we use the previous lemma; for each $s > 0$, $\sigma_n(s)$ is a Cauchy sequence by lemma, converging to $c'(s)$. $s \mapsto c'(s)$ is the geodesic ray we are after. See Figure 53. □

11.2. Cone topology on $\overline{X} = X \cup \partial X$.

Fix $x_0 \in X$, $B[x_0, r]$ system of closed balls. Let p_r be the projection onto $B[x_0, r]$.

Then, $p_r \circ p_{r'} = p_r$ if $r' \geq r$ (see Figure 54). Consider $\varprojlim B[x_0, r]$ with inverse limit topology.

Points in this space: maps $c : [0, \infty) \to X$ such that if $r' \geq r$, then $p_r(c(r')) = c(r)$. There are two types:

- $c(r') \neq c(r)$ if $r' \neq r$, c is a geodesic ray issuing from x_0 (points in ∂X).

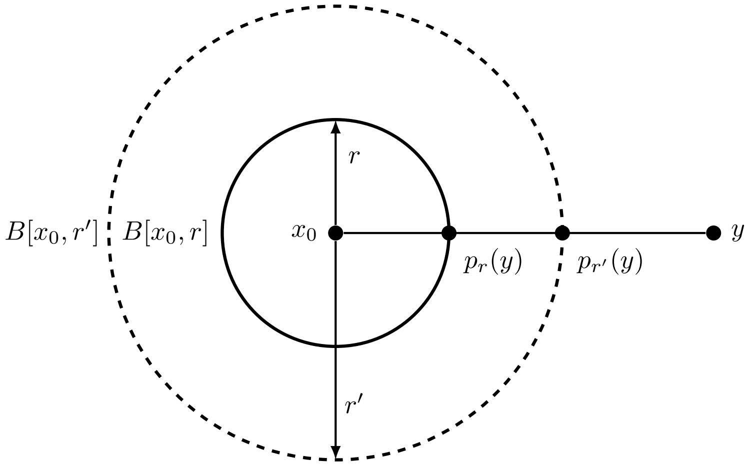


FIGURE 54. Projection for Cone topology

- There is a minimum $r_0 \geq 0$ such that $c(r) = c(r_0)$ for all $r \geq r_0$ (points in X). The restriction to $[0, r_0]$ is the geodesic segment $[c(0), c(r_0)]$.

There is a natural bijection $\phi(x_0) : X \to \varprojlim B[x_0, r]$. Topology: $\mathscr{T}(x_0)$ by requiring that $\phi(x_0)$ is a homeomorphism. ∂X is the *visual boundary*.

- $\mathscr{T}$ is independent of our chosen x_0 (follows from our proposition from previous subsection).
- Explicit nhood basis for the cone topology. Given a geodesic ray c for $p \in \partial X$, $U(c, r, \epsilon) = \{x \in X \mid d(x, c_0) > r, d(p_r(x), c(t)) < \epsilon\}$.

Corollary 11.4. *Let γ be an isometry of a complete* CAT(0) *space X. The natural extension of γ to $\overline{X}$ is a homeomorphism.*

Proof. Since the Cone topology is independent of our base point, $\varprojlim B[x_0, r] \cong \varprojlim B[\gamma x_0, r]$. □

Example 11.5. X is a complete n-dimensional Riemannian manifold of non-positive curvature. Then, $\partial X = S^{n-1}$. Also, $\mathbb{H}^n$. If X is an $\mathbb{R}$-tree, boundary X is an inverse limit of spaces $S_r(x)$ that are totally disconnected, so it must be totally disconnected. Special case: simplicial tree valency 3 everywhere, obtain a Cantor set. An example of an $\mathbb{R}$-tree: $\mathbb{R}^2$ with taxi cab metric as seen in Figure 55.

FIGURE 55. Taxi cab

11.3. **δ-Hyperbolic spaces.** Generalise $\mathbb{H}^n$ and trees.

Definition 11.6. *Rips hyperbolicity*, if we have a geodesic triangle $T = [x, y, z]$, taking δ-nhoods of $T_1 = [x, y]$, $T_2 = [y, z]$ and $T_3 = [x, z]$, the thinness radius $\delta(T) = \max_{i=1,2,3} \left(\sup_{p \in T_i} d(p, T_{i+1} \cup T_{i+2}) \right)$. T is *δ-thin*, if $\delta(T) \leq \delta$. See Figure 56.

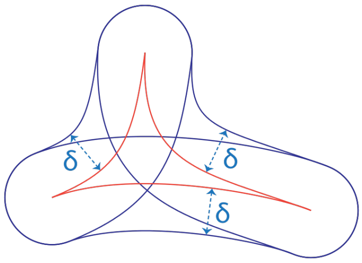


FIGURE 56. δ-Thin

Definition 11.7. A geodesic metric space is *δ-hyperbolic*, if all triangles are δ-thin.

Note. Infimum of all such δ is *hyperbolicity constant*.

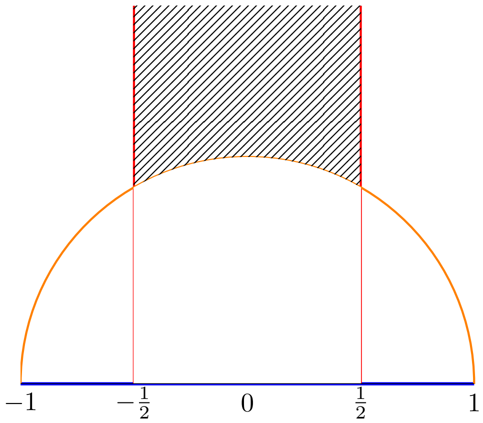


FIGURE 57. Area is $\frac{\pi}{3}$

Example 11.8. CAT(κ)-spaces for $\kappa < 0$ (comparison triangle lives in Riemannian manifold of constant negative curvature). $\mathbb{H}^2$ is δ-hyperbolic (the area of a hyperbolic triangle is bounded above by π). If the angles are α, β, γ, then $\pi - (\alpha + \beta + \gamma)$ is the area of the triangle. Related: area in Figure 57 is

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{\sqrt{1-x^2}}^{\infty} \frac{dx dy}{y^2} = \frac{\pi}{3}.$$

Another example is $\mathbb{R}$-tree, which is $\delta = 0$-hyperbolic (triangles are tripods).

Non-examples in the Euclidean plane and there sphere S^2.

Recall the Hausdorff distance is $d_H(X, Y) = \max \left\{ \sup_{x \in X} d(x, Y), \sup_{y \in Y} d(X, y) \right\}$.

Lemma 11.9 (Thin bigon property)**.** *If X is δ-hyperbolic, then all geodesics XY, ZX with $d(y, z) \leq D$ are at Hausdorff distance at most $D + \delta$ from each other.*

Theorem 11.10 (Quasi-geodesics in hyperbolic spaces). *Let X be δ-hyperbolic, c a continuous rectifiable path in X. If $[p, q]$ is a geodesic segment containing the endpoints of c, then for every $x \in [p, q]$, $d(x, \operatorname{im}(c)) \leq \delta|\log_2(\ell(c))| + 1$.*

Proof. For $\ell(c) \leq 1$, result is clear. For $\ell(c) > 1$, refer to Figure 58. Thin, so x within δ distance of one of the sides. Say if the side on the left, go to $\frac{1}{4}$, else go to $\frac{3}{4}$. Then continue in the natural dyadic rational way. □

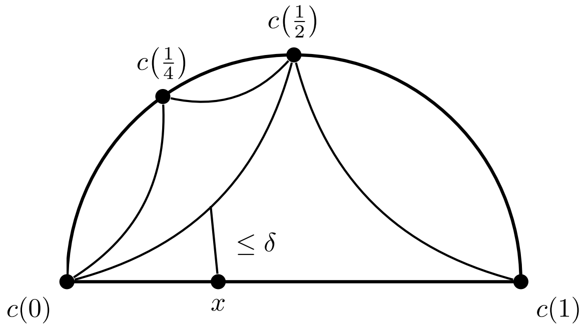


Figure 58. Bound construction

Definition 11.11. A *quasi-geodesic* is a quasi-isometric embedding of an interval (could be $[0, \infty)$), with say parameters (λ, ϵ).

Theorem 11.12 (Stability of quasi-geodesics). *For each $\delta > 0$, $\lambda \geq 1$, $\epsilon > 0$, there exists $R = R(\delta, \lambda, \epsilon)$ such that if X is a δ-hyperbolic geodesic space, c is a (λ, ϵ)-quasi geodesic in X and $[p, q]$ is a geodesic segment joining the endpoints of c, then the Hausdorff distance between $[p, q]$ and the image of c is less than R.*

Corollary 11.13. *X is δ-hyperbolic if, and only if, quasi-geodesic triangles are thin.*

Theorem 11.14. *Hyperbolicity is a quasi-isometric invariant.*

Definition 11.15. A group Γ is *hyperbolic*, if there is a finite generating set S for Γ such that Γ with the word metric of S is a hyperbolic space.

Example 11.16. Examples of hyperbolic groups include free groups and Kleinian groups.

Theorem 11.17 (Gromov). *Almost all finitely presented groups are hyperbolic.*

Theorem 11.18 (Rips). *Let Γ be a hyperbolic group. There exists a simplicial complex P, which is contractible, locally finite, and finite dimensional on which Γ acts simplicially, faithfully, and geometrically.*

Note. Finitely generated + hyperbolic implies finitely presented.

Let $n \geq 1$ be an integer. Then the *Rips complex* $P_n(\Gamma, S)$ is the simplicial complex whose k-simplicies are $(k+1)$-tuples $(\gamma_0, \ldots, \gamma_k)$ of pairwise distinct elements such that $\max_{i,j} d(\gamma_i, \gamma_j) \leq n$. We equip $P_n(\Gamma, S)$ with the weak topology.

Example 11.19. $\Gamma = \mathbb{Z}$, $S = \{-1, 1\}$. Then, $P_2(\Gamma, S)$ is as shown in Figure 59.

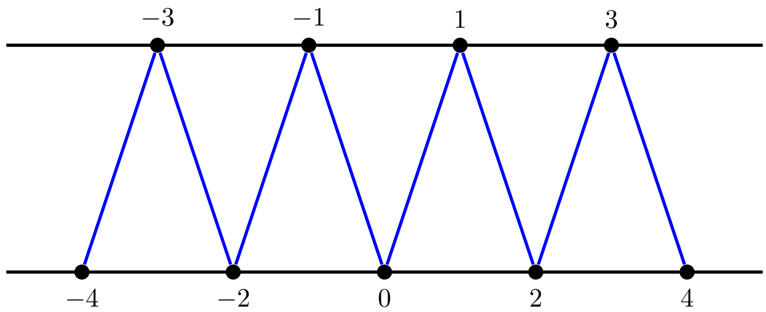


FIGURE 59. Rips complex for $\mathbb{Z}$

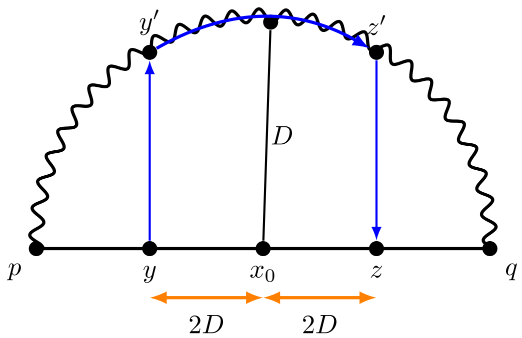


FIGURE 60. Step 2 in stability of quasi-geodesics

Theorem 11.20 (Stability of quasi-geodesics (Bridson-Haefligger III.H.1.7))**.** *For each $\delta > 0$, $\lambda \geq 1$, $\epsilon \geq 0$, there exists $R = R(\delta, \lambda, \epsilon) \geq 0$ such that if X is δ-hyperbolic, c is a (λ, ϵ) quasi-geodesic, $[p, q]$ is a geodesic segment joining the endpoints of c, then the Hausdorff distance between $[p, q]$ and the image of c is at most R.*

Proof. Step 1: Taming the geodesic. We transform the quasi-geodesic c to a continuous one c' by considering $\mathbb{Z} \cap [p, q]$.

Step 2 (see Figure 60): Let D be the maximal distance from $[p, q]$ to $\operatorname{im}(c')$. Path from y to z "blue path" γ; distance from x_0 does not enter the open ball $B(x_0, D)$. Consider $\ell(\gamma)$. We have

$$d(y', z') \leq d(y', y) + d(y, z) + d(z, z') \leq 6D.$$

So, $\ell(\gamma) \leq 6DK_1 + K_2 + 2D$ as it is a quasi-geodesic. It follows $D \leq \delta|\log_2(\ell(\gamma))| + 1$. Hence, $D \leq D_0$ where D_0 is a global constant.

Potential problem: the quasi-geodesic could be close to $[p, q]$ at start and end but mid-way have large distance. There exists $w \in [p, q]$, $t \in [a, a']$, $t' \in [b', b]$ such that $d(w, c(t)) \leq D_0$ and $d(w, c(t')) \leq D_0$. Need to bound $\ell\Big(c|_{[t,t']}\Big) \leq 2K_1 D_0 + K_2$. So, $d(c(t), c(t')) \leq 2D_0$. This follows from connectedness. □

Note. There is a converse to the stability of quasi-geodesics: Let X be a geodesic space. If X has a divergence function e such that $\lim_{n\to\infty} \inf \frac{e(n)}{n} = \infty$, then X is δ-hyperbolic.

We say c is a *k-local geodesic*, if $d(c(t), c(t')) = |t - t'|$ for each $t, t' \in [a, b]$ with $|t - t'| \leq k$.

Theorem 11.21. *k-local geodesics are quasi-geodesics.*

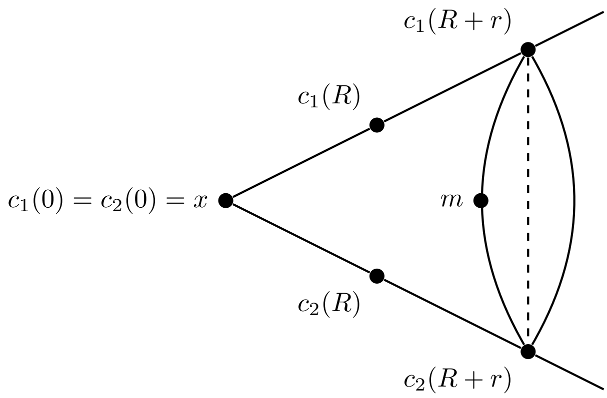


FIGURE 61. Divergence of geodesics

Definition 11.22 (Divergence of geodesics (see Figure 61))**.** A map $e : \mathbb{N} \to \mathbb{R}$ is a *divergence function* for X, if for each $R, r \in \mathbb{N}$, if $d(c_1(R), c_2(R)) > e(0)$ then any path connecting $c_1(R + r)$ and $c_2(R + r)$ outside $B(x, R + r)$ must have length at least $e(R)$.

Proposition 11.23 (Bridson-Haefligger Page 413)**.** *If X is hyperbolic, then X has an exponential divergence function.*

Proof. Assume $d(c_1(R), c_2(R)) > 3\delta$, then show midpoint m. □

12. Profinite Rigidity

Given a group Γ, let $\mathcal{C}(\Gamma)$ denote the set of isomorphism classes of finite groups that are quotients of Γ. Equivalently, by the First Isomorphism Theorem, $\mathcal{C}(\Gamma)$ is the set of isomorphism classes of finite groups that are homomorphic images of Γ. In this section, we consider profinite rigidity, which aims to answer the following.

Question 12.1. To what extent can we determine a group Γ from its set of finite quotients $\mathcal{C}(\Gamma)$?

We must place restrictions on Γ, as there are even finitely presented infinite groups with no finite quotients at all, such as

$$G_4 = \left\langle \alpha, \beta, \gamma, \delta \;\middle|\; \beta\alpha = \alpha^2\beta, \gamma\beta = \beta^2\gamma, \delta\gamma = \gamma^2\delta, \alpha\delta = \delta^2\alpha \right\rangle$$

and

$$B_3 = \left\langle a, b, c, d \;\middle|\; ba^2 = a^3b, dc^2 = c^3d, [a, b] = d, [c, d] = b \right\rangle.$$

There does not even exist an algorithm determining whether a finitely presented group has finite quotients.

Definition 12.2. A group Γ is *residually finite*, if for each $\gamma \in \Gamma \setminus \{1\}$, there exists a group homomorphism $\pi : \Gamma \to$ Finite, such that $\pi(\gamma) \neq 1$.

Note. Equivalently, Γ is residually finite if $\bigcap_{N \lhd \Gamma | [\Gamma : N] < \infty} N = 1$. Equivalently, G is residually finite, if for each $g \neq 1 \in G$ there exists $K \leq G$ of finite index such that $g \notin K$.

Example 12.3. Examples of residually finite groups include finite groups, finitely generated nilpotent groups, finitely generated linear groups, and the fundamental group of compact 3-manifolds.

Proposition 12.4. *A finitely generated group has only finitely many subgroups of a given index k.*

Proof. □

The core, $\mathrm{Core}(K) = \bigcap_{g\in G} gKg^{-1}$ is a normal subgroup of still finite index.

Lemma 12.5. *Let G be a finitely generated group. Then,*

(1) *if G is residually finite and $H \leq G$, then H is residually finite;*
(2) *if $H \leq G$ is residually finite and $H \leq G$ with finite index, then G is residually finite.*

Proposition 12.6. *Finitely generated abelian groups are residually finite.*

Proof. □

Theorem 12.7. *If G is a finitely generated linear group of characteristic 0, say $G \leq \mathrm{GL}(n, \mathbb{C})$ for some n, then G is residually finite.*

Proof. First case: $G \leq \mathrm{GL}(n, \mathbb{Z})$, suffices to show $\mathrm{GL}(n, \mathbb{Z})$ is residually finite by previous Lemma (1). Consider the homomorphism $\phi_p : \mathrm{GL}(n, \mathbb{Z}) \to \mathrm{GL}(n, \mathbb{Z}/p\mathbb{Z})$. Choose prime p larger than each of the entries in given non-trivial matrix.

General case: $G \leq \mathrm{GL}(n, \mathbb{C})$ finitely generated. May take $G \leq \mathrm{GL}(n, R)$ for ring R, then apply same argument as above. □

Definition 12.8. We say a group G is *Hopfian*, if every epimorphism $\phi : G \to G$ is a monomorphism.

Note. A non-example is $(\mathbb{R}, +)$ (i.e., it is non-Hopfian).

Lemma 12.9. *Finitely generated, residually finite groups are Hopfian.*

Proof. To derive a contradiction, suppose G is a non-Hopfian, finitely generated, residually finite group. Let $\phi : G \to G$ be an epimorphism. Suppose that $\ker\phi \neq 1$. Let $g \in \ker\phi$. By residually finiteness, there exists normal N of G such that $\psi : G \to K = G/N$ where $\psi(g) \neq 1$. Claim: there are only finitely many homomorphisms from $G \to K$. Claim follows from the fact there are finitely many generators and K is finite. Say $\alpha_1, \dots, \alpha_N : G \to K$ are all of the homomorphisms. Consider $\alpha_1 \circ \phi$, ..., $\alpha_N \circ \phi$. If $\alpha_i \circ \phi = \alpha_j \circ \phi$, then $\alpha_i = \alpha_j$ because ϕ is surjective. Hence, $\psi = \alpha_i \circ \phi$ for some i. However, $1 \neq \psi(g) = (\alpha_i \circ \psi)(g) = 1$, a contradiction. □

If Γ is finitely generated and residually finite, it turns out one can recover any finite portion of its Cayley graph and multiplication table by examining its set of finite quotients. Therefore, it is natural to ask whether we can determine finitely generated, residually finite groups Γ up to isomorphism by its set of finite quotients $\mathcal{C}(\Gamma)$.

We answer this via profinite rigidity. A rigidity theorem typically has that if two objects are equivalent in a weaker sense, it forces them to be equivalent in a stronger sense (such as the Mostow Rigidity Theorem).

Theorem 12.10 (Mostow's Rigidity Theorem)**.** *Suppose M and N are complete finite-volume hyperbolic manifolds of dimension $n \geq 3$. If there exists an isomorphism $f : \pi_1(M) \to \pi_1(N)$, then it is induced by a unique isometry from M to N.*

In order to define profinite rigidity, we must go over profinite groups.

Definition 12.11 (Axiomatic)**.** We say a topological group G is a *profinite group*, if G is compact, totally disconnected, and Hausdorff.

Example 12.12. Let G be a finite group endowed with the discrete topology. Then, G is a profinite group.

With this axiomatic definition, it is not clear how to construct profinite groups. We now go over a constructive definition, which makes use of inverse limits.

Definition 12.13. A *directed set*, is a partially ordered set $(I, \leq)$, with the property that if $i, j \in I$, then there exists $k \in I$ such that $i, j \leq k$.

Definition 12.14. Let $(I, \leq)$ be a directed set. An *inverse system*, denoted by $\langle X_i, f_{ij}, I\rangle$, is a family of spaces X_i and a family of maps $f_{ij} : X_i \to X_j$ whenever $i \geq j$, such that

- $f_{ii} = \mathrm{id}_{X_i}$; and
- $f_{jk} \circ f_{ij} = f_{ik}$ whenever $i \geq j \geq k$.

The *inverse limit*, is the set

$$\varprojlim X_i := \left\{ (x_i) \in \prod_{i \in I} X_i \;\middle|\; f_{ij}(x_i) = x_j, \forall i \geq j \right\},$$

equipped with the relative product topology.

Definition 12.15 (Constructive)**.** A *profinite group*, is a topological group G isomorphic to the inverse limit of an inverse system of discrete finite groups (and the family of maps are homomorphisms).

Example 12.16. The group of p-adic integers $\mathbb{Z}_p$ under addition is a profinite group, where $\mathbb{Z}_p \cong \varprojlim \mathbb{Z}/p^n\mathbb{Z}$. Concretely, take $p = 3$. Consider

$$\mathbb{Z}/3\mathbb{Z} \longleftarrow \mathbb{Z}/9\mathbb{Z} \longleftarrow \mathbb{Z}/27\mathbb{Z} \longleftarrow \mathbb{Z}/81\mathbb{Z} \longleftarrow \dots,$$

where $(1, 1, 1, \dots), (1, 4, 22, \dots) \in \varprojlim \mathbb{Z}/3^n\mathbb{Z}$, yet $(1, 2, 3, 4, \dots) \notin \varprojlim \mathbb{Z}/3^n\mathbb{Z}$.

Definition 12.17. The *profinite completion* of a group Γ, denoted $\widehat{\Gamma}$, is the inverse limit of $\langle \Gamma/N_i, \phi_{ij}, I\rangle$, where N_i are normal subgroups of Γ of finite index, $i \geq j$ if, and only if, $N_i \leq N_j$, and for $i \geq j$, $\phi_{ij} : \Gamma/N_i \to \Gamma/N_j$ is defined by $\phi_{ij}(gN_i) = gN_j$. That is to say,

$$\widehat{\Gamma} := \varprojlim \Gamma/N_i \qquad [\Gamma : N_i] < \infty.$$

Lemma 12.18. *The inclusion map $\iota : \Gamma \to \widehat{\Gamma}$ defined by $g \mapsto (gN_i)$ is one-to-one if, and only if, Γ is residually finite.*

Note. Each finite group Γ/N_i is equipped with the discrete topology, and $\widehat{\Gamma} \subseteq \prod_{i \in I} \Gamma/N_i$ with the subspace topology. It turns out that $\iota(\Gamma)$ is dense in $\widehat{\Gamma}$. Other properties include, for $\Delta \leq \Gamma$:

- $\Delta \lhd \Gamma$ if, and only if, $\overline{\iota(\Delta)} \lhd \widehat{\Gamma}$.
- $[\Gamma : \Delta] = [\widehat{\Gamma} : \overline{\iota(\Delta)}]$.
- If $\Delta \lhd \Gamma$, then $\Gamma/\Delta \cong \widehat{\Gamma}/\overline{\iota(\Delta)}$.

Theorem 12.19 (Nikolov and Segal)**.** *Every subgroup of finite index in $\widehat{\Gamma}$ is open.*

Theorem 12.20 (Nikolov and Segal)**.** *If Γ is finitely generated, then every group homomorphism from $\widehat{\Gamma}$ to a profinite group is continuous.*

With the following observation, we can translate our question on finite quotients to that of profinite completions.

Theorem 12.21. *For finitely generated groups* Γ_1 *and* Γ_2*,* $\mathcal{C}(\Gamma_1) = \mathcal{C}(\Gamma_2)$ *if, and only if,* $\widehat{\Gamma}_1 \cong \widehat{\Gamma}_2$*.*

Note. By Theorem 12.20, it suffices to show $\widehat{\Gamma}_1$ and $\widehat{\Gamma}_2$ are isomorphic as groups, as it will force the isomorphism to become a homeomorphism (and thus they will be isomorphic as topological groups).

Definition 12.22. We say a finitely generated, residually finite group Γ is *profinitely rigid in the absolute sense*, if for each finitely generated, residually finite group Δ, $\widehat{\Gamma} \cong \widehat{\Delta}$ implies $\Gamma \cong \Delta$.

Example 12.23. Finite groups are profinitely rigid. For suppose G is a finite group, and Γ is a finitely generated, residually finite group such that $\widehat{G} \cong \widehat{\Gamma}$. Since G is finite, it is a profinite group, implying $G \cong \widehat{G}$. Hence, $\widehat{\Gamma}$ is finite. As Γ is residually finite, by Lemma 12.18, the inclusion map $\iota : \Gamma \hookrightarrow \widehat{\Gamma}$ is one-to-one. Therefore, Γ is finite. Since $\iota(\Gamma)$ is dense in $\widehat{\Gamma}$, it follows ι is onto. Thus, ι is an isomorphism, implying

$$G \cong \widehat{G} \cong \widehat{\Gamma} \cong \Gamma,$$

and we are done. Infinite groups need infinite finite quotients.

Example 12.24. Finitely generated abelian groups are profinitely rigid. For example, we show $\mathbb{Z}$ is profinitely rigid. To this end, suppose Γ is a finitely generated, residually finite group such that $\widehat{\Gamma} \cong \widehat{\mathbb{Z}}$. Then, Γ is abelian, since if $[\gamma, \delta] \neq 1$, then $[\phi(\gamma), \phi(\delta)] \neq 1$ in some finite quotient $\phi : \Gamma \to Q$ (because Γ is residually finite). That is to say, a finite quotient would therefore not be Abelian (but all finite quotients of $\mathbb{Z}$ are Abelian). By the Basis Theorem from MATHS720, since G is a finitely generated Abelian group, $G = \mathbb{Z}_{d_1} \oplus \ldots \oplus \mathbb{Z}_{d_k} \oplus \mathbb{Z}^s$. We deduce $G = \mathbb{Z}^s$ for some s, since otherwise $\mathbb{Z}_{d_1} \oplus \mathbb{Z}_{d_1}$ is a finite quotient of G, which isn't one of $\mathbb{Z}$ (finite quotients of $\mathbb{Z}$ are $\mathbb{Z}/n\mathbb{Z}$). Now, if $s > 1$, by similar argument, can get a finite quotient of G not one of $\mathbb{Z}$.

Example 12.25. $N \times \mathbb{Z}$, where N is finite, is profinitely rigid. However, Baumslag in 1974 showed there exists metacyclic groups $\Gamma = (\mathbb{Z}/25\mathbb{Z}) \rtimes_\alpha \mathbb{Z}$ and $\Delta = (\mathbb{Z}/25\mathbb{Z}) \rtimes_\beta \mathbb{Z}$ such that $\widehat{\Gamma} \cong \widehat{\Delta}$, but $\Gamma \ncong \Delta$. Using multiplicative notation, $\alpha, \beta \in \mathrm{Aut}\,(\mathbb{Z}/25\mathbb{Z})$ such that $\alpha(x) = x^6$ and $\beta(x) = x^{11}$.

Not necessarily true that if G and H are both profinitely rigid, then $G \times H$ is profinitely rigid. In fact, if this were the case, it would solve an open problem. We know $F_2 \times F_2$ is not profinitely rigid, however, the following is an open question.

Question 12.26 (Remeslennikov)**.** Are finitely generated free groups of rank at least 2 profinitely rigid?

One starts to see positive results when discussing profinite rigidity in a relative sense, i.e., comparing our group with a subset of finitely generated, residually finite groups.

Theorem 12.27 (Bridson-Conder-Reid, 2016)**.** *Let* Γ *be a finitely generated Fuchsian group, i.e., a lattice in* $\mathrm{PSL}\,(2, \mathbb{R})$*, and let* Λ *be a lattice in a connected Lie group. If* $\widehat{\Gamma} \cong \widehat{\Lambda}$*, then* $\Gamma \cong \Lambda$*.*

Theorem 12.28. *If* Δ_1 *and* Δ_2 *are triangle groups with the same set of finite quotients, then they are isomorphic.*

Below are examples of arithmetic Kleinian groups.

- The Bianchi group $\mathrm{PSL}\,(2, \mathbb{Z}[\omega])$ with $\omega^2 + \omega + 1 = 0$ is profinitely rigid in the absolute sense.

- Non-uniform lattice of minimal co-volume in $\mathrm{PSL}(2, \mathbb{C})$ is profinitely rigid in the absolute sense.
- The fundamental group of the Weeks manifold (the closed hyperbolic 3-manifold of minimal volume) is profinitely rigid in the absolute sense.

13. Arithmetic Kleinian Groups

(Slides from "A (Hopefully Gentle) Introduction to Arithmetic Kleinian Groups").

We recall a *Kleinian group* is a discrete subgroup of $\mathrm{PSL}(2, \mathbb{C})$. $\mathrm{PSL}(2, \mathbb{Z})$ is an example of an arithmetic Kleinian group. The group of orientation preserving isometries on $\mathbb{H}^3$ can be identified with $\mathrm{PSL}(2, \mathbb{C})$. A *hyperbolic 3-orbifold* is the quotient of $\mathbb{H}^3$ by a Kleinian group, denoted $O \cong \mathbb{H}^3/\Gamma$. If $\mathbb{H}^3/\Gamma$ is compact, we say Γ is *cocompact*. If $\mathbb{H}^3/\Gamma$ has finite volume, we say that Γ has *finite covolume*.

Two Kleinian groups Γ and Γ' are *commensurable*, if $\Gamma \cap \Gamma'$ has finite index in both Γ and Γ'. Two Kleinian groups Γ and Γ' are *commensurable in the wide sense*, if there exists $g \in \mathrm{PSL}(2, \mathbb{C})$ such that $g^{-1}\Gamma g$ and Γ' are commensurable. If $O \cong \mathbb{H}^3/\Gamma$ and $O' \cong \mathbb{H}^3/\Gamma'$, then this definition is equivalent to O and O' having a common sheeted cover.

Let $\Gamma \leq \mathrm{PSL}(2, \mathbb{C})$. Γ is *reducible*, if all the elements of Γ have a common fixed point, otherwise it is *irreducible*. The group is *elementary*, if it has finite orbit in its action on $\mathbb{H}^3 \cup \partial\mathbb{H}^3$, otherwise it is *non-elementary*. Notice that reducible groups are elementary, but that elementary groups need not be reducible. Note that elementary groups have been classified.

We now recall some field theory. We may extend the rationals by adding the root to $x^2 - 2 = 0$, obtaining $\mathbb{Q}(\sqrt{2}) := \{a + b\sqrt{2} \mid a, b \in \mathbb{Q}\}$; also, may extend the rationals further by $\mathbb{Q}(\sqrt{2}, \sqrt{3})$ (Dedekind's Theorem). Let $k \setminus \mathbb{Q}$ be a finite extension of fields (i.e., k is a field that is a finite dimensional vector space over $\mathbb{Q}$ as with previous examples). If $[k : \mathbb{Q}] = n$, then we n embeddings and the following.

- There exists $t \in k$ such that the minimal polynomial of t over $\mathbb{Q}$ has degree n and $k \cong \mathbb{Q}(t)$.
- If $t = t_1, t_2, \ldots, t_n$ are the roots of the minimal polynomial of t then the assignments $t \mapsto t_i$ induces an embedding, denoted σ_i of k into $\mathbb{C}$.
- Conversely, any embedding of k into $\mathbb{C}$ must map t to one of the t_i, and thus there are exactly n embeddings of k into $\mathbb{C}$.
- If $\sigma_i(k) \subset \mathbb{R}$ then the embedding is called a *real place*. Otherwise, if $\sigma_i(k) \not\subset \mathbb{R}$, then there exists $j \neq i$ such that $\sigma_j = \overline{\sigma_i}$ and we refer to the pair (σ_i, σ_j) as a *complex place*.
- Therefore we arrive at the formula $n = r_1 + 2r_2$ where r_1 is the number of real places and r_2 is the number of conjugate places of complex embeddings.

Theorem 13.1 (Superrigidity Theorem (Margulis))**.** *Suppose G is a semi-simple Lie group (inside some $\mathrm{SL}(n, \mathbb{R})$) of rank at least 2; Γ is an irreducible lattice in G; H is a connected, non-compact, simple subgroup of some $\mathrm{SL}(m, \mathbb{R})$ with trivial center; and $\varphi : \Gamma \to H$ is an irreducible representation such that $\varphi(\Gamma)$ is Zariski dense in H. Then, φ extends to a continuous homomorphism $\widehat{\varphi} : G \to H$.*

We define the *norm of α* to be $N_{k\setminus\mathbb{Q}}(\alpha) := \prod_{i=1}^{n} \sigma_i(\alpha)$, and the *trace of α* to be $Tr_{k\setminus\mathbb{Q}}(\alpha) := \sum_{i=1}^{n} \sigma_i(\alpha)$. Note $x^2 - (\mathrm{tr}(x))x + n(x) = 0$.

Let $\overline{\mathbb{Q}}$ be the algebraic closure of $\mathbb{Q}$ $(\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}))$. Then, $\alpha \in \overline{\mathbb{Q}}$ is an *algebraic integer*, if its minimal polynomial has entries in $\mathbb{Z}$; if $x + a = 0$, forces x to be an integer if a is an integer. Given a finite extension k of $\mathbb{Q}$, then $\mathcal{O}_k$ be the set of algebraic integers contained in k.

Recall the Modular group $\mathrm{PSL}(2,\mathbb{Z}) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PSL}(2,\mathbb{C}) \;\middle|\; a,b,c,d \in \mathbb{Z} \right\}$. This is an arithmetic Kleinian group. If $\Gamma = \mathrm{PSL}(2,\mathbb{Z})$, then $O = \mathbb{H}^3/\Gamma$ is the 2-orbifold with base space the disk and two cone points of order 2 and 3, respectively.

Let $k = \mathbb{Q}(\sqrt{-d})$, where $d > 0$, then $\mathrm{PSL}(2,\mathcal{O}_k)$ is a *Bianchi group*. These groups are the 3-dimensional analogue of the modular group. Bianchi groups provide infinitely many commensurability classes of arithmetic, Kleinian groups. All 3-dimensional cusped, arithmetic, Kleinian groups are commensurable with a Bianchi group.

A *quaternion algebra* A over a field F of characteristic $\neq 2$ is a four-dimensional algebra over the field F, with basis $\{1, i, j, k\}$ and elements $a, b \in F^*$ such that

$$i^2 = a, j^2 = b, ij = -ji = k.$$

A quaternion algebra can be described by a (non-unique) *Hilbert symbol* $(\frac{a,b}{F})$, where the entries are squares of a pair of basis elements other than 1. The usual Hamiltonian quaternions from $\mathbb{R}$, denoted $\mathbb{H}$, is obtained by $a = b = -1$.

We now consider isometries of $\mathrm{PSL}(2,\mathbb{C})$ on $\mathbb{H}^3$ (Voight Quaternion Algebras Chapter 36). Recall $\mathrm{PSL}(2,\mathbb{R})$ acts on $\mathbb{H}^2$ as the orientation-preserving isometries of the hyperbolic plane. Discrete subgroups of $\mathrm{PSL}(2,\mathbb{R})$ are Fuchsian groups, and of $\mathrm{PSL}(2,\mathbb{C})$ are Kleinian groups. The orientation preserving isometries on $\mathbb{H}^3$ is $\mathrm{PSL}(2,\mathbb{Z}[i])$, the Picard Modular Group (which is an example of a Bianchi group).

We have $\mathbb{H}^3$ embeds into the Hamiltonian quaternions $\mathbb{H}$ by $(x,y) \mapsto z = x + yj$, where $x \in \mathbb{C}$ and $y \in \mathbb{R}_{>0}$. The group action $\mathrm{SL}(2,\mathbb{C}) \times \mathbb{H}^3 \to \mathbb{H}^3$ by $(g,z) \mapsto \frac{az+b}{cz+d}$. Relates to the quaternionic projective plane $\mathbb{P}^1(\mathbb{H})$. We have $z \mapsto \frac{az+b}{cz+d}$ which can be expressed by $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} z \\ 1 \end{pmatrix} = \begin{pmatrix} az+b \\ cz+d \end{pmatrix}$. Note $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ corresponds to ∞ of Riemann sphere (division by 0).

Note. $\mathrm{SO}(2)$ is the point stabiliser of i for $\mathbb{H}^2$. So, $\mathrm{SU}(2)$ is the analogue for $\mathbb{H}^3$.

Iwasawa decomposition: $K = \mathrm{SU}(2) \cong \mathbb{H}^1$. We have $A = \left\{ \begin{pmatrix} a & 0 \\ 0 & \frac{1}{a} \end{pmatrix} \;\middle|\; a \in \mathbb{R}_{>0} \right\} \cong \mathbb{R}$ and $N = \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \;\middle|\; b \in \mathbb{C} \right\} \cong \mathbb{C}$. Then, $K \times A \times N \to \mathrm{SL}_2(\mathbb{C})$.

Theorem 13.2. *The action defined above is faithful, transitive and by isometries.*

Note. We could define this more geometrically by Poincaré extension.

We now consider Hyperbolic volume by an ideal tetrahedron (all 4 vertices are on the boundary; see Figure 62). We have $\alpha + \beta + \gamma = 2\pi$ (the angle at ∞ is 0). The Lobeschevsky function $\mathcal{L} : \mathbb{R} \to \mathbb{R}$ given by $\mathcal{L}(\theta) = -\int_0^\theta \log|2\sin(t)|dt$ and $v = \mathcal{L}(\alpha) + \mathcal{L}(\beta) + \mathcal{L}(\gamma)$. Then, $\mathcal{L}(\theta) = \frac{1}{2}\sum_{n=1}^\infty \frac{\sin(2n\theta)}{n^2}$.

Apply this to the Picard modular group $\mathrm{PSL}(2,\mathbb{Z}[i])$. The fundamental domain $D = \left\{ z = x + yj \in \mathbb{H}^3 \;\middle|\; |z|^2 \geq 1, |\mathrm{Re}(x)| \leq \frac{1}{2}, 0 \leq \mathrm{Im}(x) \leq \frac{1}{2} \right\}$. This is the 3-dimensional analogue of Figure 57.

Note $\mathrm{vol}(D) = \frac{2}{3}\mathcal{L}(\frac{\pi}{4})$. The *Dirchlet character* is

$$\chi(n) = \begin{cases} 0 & \text{if } 2 \mid n; \\ 1 & \text{if } n \equiv 1 \mod 4; \\ -1 & \text{if } n \equiv 3 \mod 4. \end{cases}$$

Then, $\mathcal{L}(\frac{\pi}{4}) = \frac{1}{2}\sum_{n=1}^\infty \frac{\sin\frac{n\pi}{2}}{n^2} = \frac{1}{2}\sum_{n=1}^\infty \frac{\chi(n)}{n^2}$. Taking $\mathcal{L}(s,\chi) = \sum_{n=1}^\infty \frac{\chi(n)}{n^s}$ we see that $\mathrm{Vol}(D) = \mathrm{Vol}(\Gamma/\mathbb{H}^3) = \frac{1}{3}\mathcal{L}(2,\chi) \approx 0.30532\ldots$

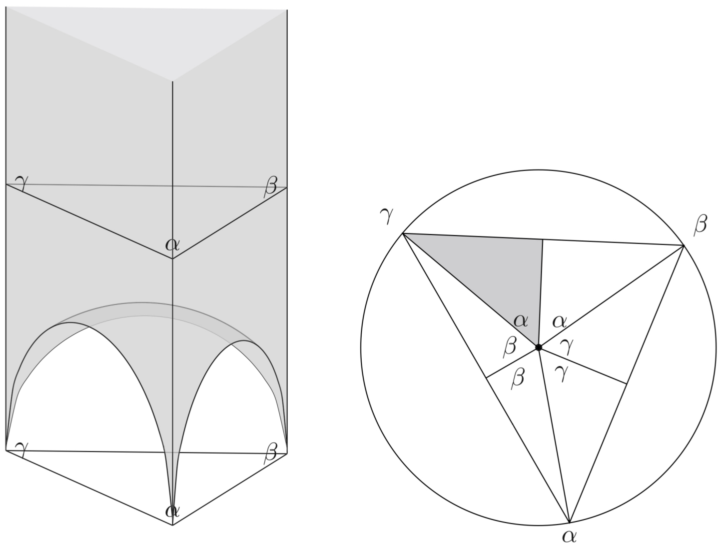


FIGURE 62. Ideal tetrahedra and its shadow in $\mathbb{C}$

14. Ultrafilters and Ultralimits

Definition 14.1. A *filter* on a given set X, is a collection $\mathscr{F} \subseteq \mathcal{P}(X)$ satisfying:

(1) If $F \in \mathscr{F}$ and $F \subseteq S \subseteq X$, then $S \in \mathscr{F}$ (i.e., $\mathscr{F}$ is upward closed).
(2) If $\mathcal{F} \subseteq \mathscr{F}$ is finite, then $\bigcap \mathcal{F} \in \mathscr{F}$ (i.e., $\mathscr{F}$ is closed under finite intersection).
(3) $\varnothing \notin \mathscr{F}$ (i.e., $\mathscr{F}$ is non-trivial).

Note. A filter gives some notion of "sufficiently large" subsets of X.

Definition 14.2. The *upwards closure* of a collection $\mathscr{A}$ of subsets of a given set X, is the collection of supersets of elements in $\mathscr{A}$. A collection of sets $\mathscr{B}$ is a *filter base on X*, if its upwards closure is a filter on X.

Proposition 14.3. *Given a set X, $\mathscr{B}$ is a filter base on X if, and only if, $\mathscr{B}$ does not contain $\varnothing$ and is closed under finite intersection.*

Proof. Suppose $\mathscr{B}$ is a filter base on X. Then, its upward closure $\mathscr{F}$ is a filter on X. Indeed, $\mathscr{B} \subseteq \mathscr{F}$, so $\varnothing \notin \mathscr{B}$ and $\mathscr{F}$ is closed under finite intersection.

Conversely, suppose $\mathscr{B}$ does not contain $\varnothing$ and is closed under finite intersection. Let $\mathscr{F}$ be the upwards closure of $\mathscr{B}$. Then, $\varnothing \notin \mathscr{F}$ and $\mathscr{F}$ is upward closed. Suppose $\mathcal{F} \subseteq \mathscr{F}$ is finite. For each $F \in \mathcal{F}$, there exists $B_F \in \mathscr{B}$ such that $B_F \subseteq F$. Let $\mathcal{B} = \{B_F \in \mathscr{B} \mid F \in \mathcal{F}\}$. Then, $\bigcap \mathcal{B} \in \mathscr{B}$. Since $\bigcap \mathcal{B} \subseteq \bigcap \mathcal{F}$, it follows $\bigcap \mathcal{F} \in \mathscr{F}$. Thus, $\mathscr{F}$ is a filter on X, which implies $\mathscr{B}$ is a filter base on X. □

Note. Bases for ultrafilters are of particular interest.

Definition 14.4. An *ultrafilter on X* is a maximal filter $\mathscr{F}$ on X, that is, no more elements can be added to $\mathscr{F}$ or it will become $\mathcal{P}(X)$ and therefore not a filter. Equivalently, for each $A \subseteq X$ either $A \in \mathscr{F}$ or $X \setminus A \in \mathscr{F}$.

Ultrafilters are usually distinguished between *principal* and *non-principal* (or free) ultrafilters. A principal ultrafilter is one with a minimal element, equivalently it is generated by a single element (called the *principal element*).

An important result (when assuming AC) is the following.

Lemma 14.5. *Any filter on a set X is contained in an ultrafilter.*

Proof. Suppose $\mathscr{F}$ is a filter on X. Let $\mathscr{U}$ be the collection of all filters which contain $\mathscr{F}$. Let $\mathscr{C}$ be an ascending chain in $\mathscr{U}$. Then, $\bigcup\mathscr{C}$ is also a filter, and an upper bound to $\mathscr{C}$. By Zorn's Lemma, the result follows, as there is a maximal element in $\mathscr{U}$. $\square$

If $\mathscr{U}$ is an ultrafilter base on ω, then $\aleph_0 < |\mathscr{U}| \leq \mathfrak{c}$, where the latter inequality can be made strict under certain models of set theory.

If $\mathscr{F}$ is a filter, then $\mathscr{F} \cup \{\varnothing\}$ is a topology. If $\mathscr{F}$ is an ultrafilter, then $\mathscr{F} \cup \{\varnothing\}$ is connected (no non-trivial clopen sets) and each subset is either closed or open.

Definition 14.6. A filter $\mathscr{F}$ on a topological space X *converges to $x \in X$*, if every nhood of x is contained in $\mathscr{F}$.

Proposition 14.7. *A topological space X is compact if, and only if, every ultrafilter converges to some $x \in X$.*

Proof. Suppose X is compact. To derive a contradiction, suppose $\mathscr{U}$ is an ultrafilter which does not converge. For each $x \in X$, it follows there exists an open nhood U_x of x such that $U_x \notin \mathscr{U}$. Observe $\{U_x \mid x \in X\}$ is an open covering of X, so by compactness there exists a finite subcover $\{U_{x_1}, \ldots, U_{x_n}\}$ for X. Then, $X = \bigcup_{i=1}^n U_{x_i}$, implying $\varnothing = \bigcap_{i=1}^n X \setminus U_{x_i}$ by De Morgan's Laws. However, $U_{x_i} \notin \mathscr{U}$ for each $i \in \{1, \ldots, n\}$, so by maximality $X \setminus U_{x_i} \in \mathscr{U}$ for each $i \in \{1, \ldots, n\}$. As $\mathscr{U}$ is closed under finite intersection, this implies $\varnothing \in \mathscr{U}$, a contradiction. Thus, every ultrafilter on X converges.

Conversely, suppose every ultrafilter converges to some $x \in X$. Recall X is compact iff for every collection of closed subsets of X with the finite intersection property has non-empty intersection. Suppose $\mathscr{A}$ is a collection of closed sets of X with the finite intersection property. Then, $\mathscr{A}$ is a filter base on X. Let $\mathscr{F}$ be the filter generated by $\mathscr{A}$. There is an ultrafilter $\mathscr{U}$ which contains $\mathscr{F}$ with $\mathscr{U} \to x \in \bigcap\mathscr{A}$. $\square$

The *Hausdorff metric* between two subsets A and B is the supremum distance from a point in one subset to the other subset, that is,

$$d_H(A, B) = \max\left\{\sup_{a\in A} d(a, B), \sup_{b\in B} d(A, b)\right\}.$$

It is a definition of how far apart two subsets are from one another.

The *Gromov-Hausdorff metric* extends this to compare how close two compact metric spaces are to being isometric. Note this metric is complete, path-connected and geodesic. We define $d_{\mathrm{GH}}(X, Y)$ to be the infimum of $d_H(f(X), g(Y))$ for all compact metric spaces M and isometries $f : X \to M$ and $g : Y \to M$, i.e.,

$$d_{\mathrm{GH}}(X, Y) = \inf_{f,g,M} d_H(f(X), f(Y)).$$

This induces a topology on the set of compact metric spaces modulo isometry, that is the *Gromov-Hausdorff space*.

Let $\mathscr{U}$ be a non-principal filter on ω. Let (X, d) be a metric space, $\langle x_n \mid n \in \omega\rangle$ a sequence of points in X, and $x \in X$ a point. Then, x is the *$\mathscr{U}$-limit of $\langle x_n \mid n \in \omega\rangle$*, written $\mathscr{U}\text{-}\lim\langle x_n\rangle$ or $\mathscr{U}\text{-}\lim_n x_n$, if for every nhood N of x, the set $\{i \in \omega \mid x_i \in N\}$ is in $\mathscr{U}$.

$\mathscr{U}$-limits are unique, and when a conventional limit is defined, it is equal to the $\mathscr{U}$-limit.

Proposition 14.8. *If (X, d) is compact, then every $\mathscr{U}$-limit in X converges.*

Proof. Let (X, d) be compact, and $\mathscr{U}$ be a non-principal ultrafilter on ω. Let $\langle x_i \mid i \in \omega\rangle$ be a sequence in X that does not converge. To derive a contradiction, suppose there exists $\mathscr{U}$-limit that does not converge. For each $x \in X$, let U_x be an open nhood of x such that $\widetilde{U_x}$ (indices) is not in $\mathscr{U}$. Since $\{U_x \mid x \in X\}$ is an open cover of X, by compactness there is a finite subcover, $\{U_{x_1}, \dots, U_{x_n}\}$. Then, as before, $\varnothing = \bigcap_{i=1}^n X \setminus U_{x_i}$, which implies $\varnothing = \bigcap_{i=1}^n \omega \setminus \widetilde{U_{x_i}}$ is in $\mathscr{U}$ (a contradiction). □

We want to generalise the Gromov-Hausdorff convergence to wider classes of spaces. Suppose we have a sequence $\langle X_n \mid n \in \omega\rangle$ with metrics $d_{X_n} : X_n \times X_n \to [0, \infty]$ of (not necessarily compact) metric spaces. For a non-principal ultrafilter $\mathscr{U}$ we define the *ultralimit* $X_{\mathscr{U}}$ (sometimes written X_∞ or $\mathscr{U}\text{-lim}_n X_n$) as follows. Let Seq be the space of sequences $\langle x_n \mid n \in \omega\rangle$ with $x_n \in X_n$ for each $n \in \omega$. Given $x = \langle x_n \mid n \in \omega\rangle, y = \langle y_n \mid n \in \omega\rangle \in \text{Seq}$ we define their distance as

$$d_{\mathscr{U}}(x, y) = \mathscr{U}\text{-lim}_n d_{X_n}(x_n, y_n).$$

Now, if we identify all points with zero distance we obtain a metric space, which is $X_{\mathscr{U}}$. If we restrict to points having finite distance from some other fixed point, then we obtain a based (or pointed) ultralimit, given by $X^0_{\mathscr{U}}$.

Proposition 14.9. *Let each X_n be compact, and converge in the Gromov-Hausdorff topology to a compact metric space X. Then, for every non-principal ultrafilter $\mathscr{U}$, X is isometric to $X_{\mathscr{U}}$.*

Now we can use this stronger notion of convergence to talk about any metric space, and know that for our already established results they still hold without needing to worry about the ultrafilter.

Some properties of ultralimits:

- A based ultralimit of a metric space is complete.
- A based ultralimit of geodesic metric spaces is geodesic.
- A based ultralimit of $\text{Cat}(\kappa)$ spaces where $\kappa \leq 0$ is $\text{Cat}(\kappa)$.
- A based ultralimit of $\text{Cat}(\kappa_n)$ spaces where $\kappa_n \to -\infty$ is an $\mathbb{R}$-tree.

$\mathbb{R}$-building (generalisation of an $\mathbb{R}$-tree), we can metric complete usual way (not necessarily). Instead, take ultraproduct to obtain complete $\mathbb{R}$-building. May embed by $x \mapsto (x, x, \dots)$. If a group $G \curvearrowright X$, then obtain natural action by $g \cdot (x_n)_{n\in\omega} = (g \cdot x_n)_{n\in\omega}$.

15. Fundamental Domains

From Beardon Chapter 9 (The Geometry of Discrete Groups).

Let G be a Fuchsian group acting on the Poincaré disc Δ (or upper half plane $\mathbb{H}^2$). A *fundamental set* for G is a subset F of Δ which contains exactly one point from each orbit in Δ. Thus no two distinct points in F are G-equivalent and $\bigcup_{g\in G} g(F) = \Delta$. The Axiom of Choice guarentees the existence (but little else) of a fundamental set for G. A fundamental domain is a domain which, with part of its boundary, forms a fundamental set for G.

A subset D of the hyperbolic plane is a *fundamental domain* for a Fuchsian group G, if

(1) D is a domain (non-empty, connected, open);
(2) there is some fundamental set F with $D \subseteq F \subseteq \widetilde{D}$; and
(3) the hyperbolic area of the boundary is 0, i.e., $\text{h-area}(\partial D) = 0$.

Note. Let E be a subset of the hyperbolic plane. Then, $\widetilde{E}$ denotes the relative closure of E with respect to the hyperbolic plane.

Theorem 15.1. *If F_1, F_2 are measurable sets for G, then* $\text{h-area}(F_1) = \text{h-area}(F_2)$. *Let F_0 be a measurable fundamental set for subgroup G_0 of index k in G. Then,* $\text{h-area}(F_0) = k\text{h-area}(F_1)$.

Need for locally finite fundamental domain: Problematic example: Take $\mathbb{C}^*$ generate G by $z \mapsto 2z$. Then, $\mathbb{C}^*/G$ is a torus. Although not Fuchsian, other problematic examples are built off this (which are Fuchsian). See Figure 63 for $\widetilde{D}/G$ not compact.

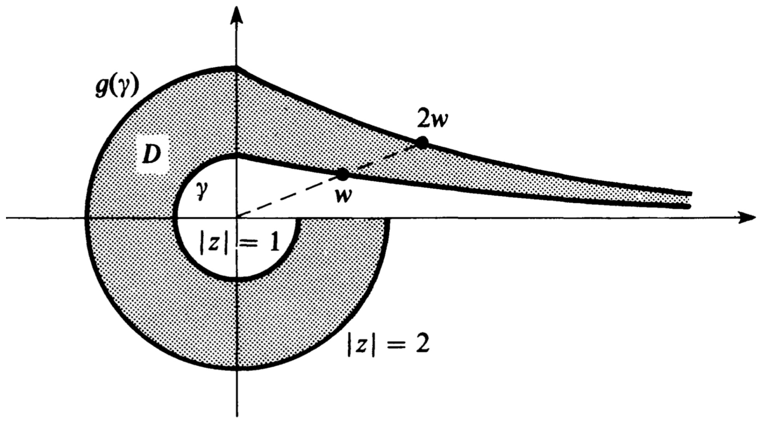


Figure 63. $\widetilde{D}/G$ not compact

Consider the following commutative diagram.

$$\begin{array}{ccc} \widetilde{D} & \overset{\iota}{\hookrightarrow} & \Delta \\ {\scriptstyle\widetilde{\pi}}\downarrow & & \downarrow{\scriptstyle\pi} \\ \widetilde{D}/G & \underset{\theta}{\longrightarrow} & \Delta/G \end{array}$$

We would like θ to be a homeomorphism.

A fundamental domain D for G is loocally finite iff each compact set of Δ meets only finitely many G-images of $\widetilde{D}$.

Theorem 15.2. *A domain is locally finite iff θ is a homeomorphism.*

Proof. We firstly prove θ is always continuous. Take open $A \subseteq \Delta/G$. Then, $\widetilde{\pi}^{-1}\big(\theta^{-1}(A)\big) = \pi^{-1}(A) \cap \widetilde{D}$ is open. So, $\theta^{-1}(A)$ is open. □

16. A Tits Alternative for $\widetilde{A}_2$ buildings

16.1. **Tits alternative for CAT(0) groups.**

Theorem 16.1 (Tits 1972)**.** *Every finitely generated linear group is either virtually solvable or contains a non-abelian free group.*

Note. Underlying theme is ping-pong lemma.

The following is an open question by Bridson and Bestvina.

Question 16.2. Do all CAT(0) groups satisfy a Tits alternative?

Recall CAT(0) groups are groups acting geometrically (properly discontinuously and co-compactly) on a CAT(0) space.

The answer is yes in several cases:

Noskov-Vinberg:	for subgroups of Coxeter groups, 2002.
Sageev-Wisel/Cature-Sagert:	for CAT(0) cube complexes.

Recently: Martiti-Przytycki in 2020/21: certain classes of Artin groups ($x_i x_j x_i = x_j x_i x_j$). Act on linear complexes.

Theorem 16.3 (Osajde-Przytycki)**.** *Let G be a finitely generated group acting almost freely (with uniformly bounded finite all stabilisers). On a 2-dimensional* CAT(0) *complex. Then G is either virtually $\mathbb{Z}$, virtually $\mathbb{Z}^2$ or contains a non-abelian free group.*

Note. Why not $\mathbb{Z}^n$, $n \geq 3$? Answer: Flat Torus Theorem.

16.2. **Tits alternative for groups acting on $\mathbb{R}$-buildings.** Firstly, we consider $\mathbb{R}$-trees: Metric space T is called an $\mathbb{R}$-*tree*, if for any 2 points $x, y \in T$, there exists a unique geodesic $\gamma : [0, d(x,y)] \to T$ such that $\gamma(0) = x$, $\gamma(d(x,y)) = y$; and if $0 < r < s$, and if $\gamma : [0,s] \to T$ is an injection such that $\gamma|_{[0,r]}$ and $\gamma|_{[r,s]}$ are geodesics, then γ is a geodesic. Example: asymptotic cones of symmetric spaces, used in Brownian motion and phylogentics.

We now consider $\mathbb{R}$-buildings: Definition is reminiscent of Riemannian manifolds. An example is tesselation of plane equaliteral triangles of type $\widetilde{A}_2$. Definition: An $\mathbb{R}$-*building*, is a metric space X together with a collection $\mathscr{A}$ of charts $\mathbb{R}^n \to X$ such that:

($EB1$) For each $\varphi \in \mathscr{A}$, $w \in W\mathbb{R}^n$, $\varphi \circ w \in \mathscr{A}$.
($EB2$) For any two points, there is an apartment containing them.
($EB3$) Charts are W-compatible, i.e., $\gamma = \varphi^{-1}\psi(\mathbb{R}^n)$ is convex (in Euclidean sense), there exists $w \in W\mathbb{R}^n$ such that $\varphi \circ w|_\gamma = \psi_1 \mid \gamma$.
($EB4$) Given two sectors, two subsectors contained in an apartment.
($EB5$) If $A \subset X$ is an affine apartmnent, $x \in A$ point, there exists 1-Lipschtiz retraction $\rho : X \to A$ with $\rho^{-1}(x) = X$.

Theorem 16.4 (Tits)**.** *Every affine $\mathbb{R}$-building of dimension at least 3 is Bruhat-Tits.*

Note. Bruhat-Tits means "of algebraic origin".

Remarks:

- Non-discrete buildings are not necessarily metrically complete even when they are Bruhat-Tits (Jeroen, Steinke, Struyve, Martson)
- Completion of a building might not be a building.
- Ultraproduct of a building is a complete building (X sits inside X^{ultra} as isometric copy).

16.3. **A tits alternative $\widetilde{A}_2$-buildings.**

Theorem 16.5 (Le Bors, Le'curex, Jeroen S.)**.** *A countable group with a non-elementary isometric action on $\widetilde{A}_2$-building contains a non-abelian free subgroup.*

Elementary isometric action: bounded orbit on X, finite orbit on ∂X.

Corollary 16.6. *A countable subgroup acting properly on an $\widetilde{A}_2$ building is either virtually $\mathbb{Z}$, virtually $\mathbb{Z}^2$, or contains a non-abelian free subgroup.*

16.4. **Random walks to produce hyperbolic elements.** Strongly regular hyperbolic elements: recall hyperbilic elements have translation axis. Strongly regular: Min(g) is a unique apartment (in general position, i.e., not one of the tesselation lines). We have attracting and repelling fixed points.

Random walks of groups: μ is probability measure on G whose symmetric support generates G. (z_n) associated random walk. Pick random elements of low μ. Form product $z_n = g_1 \dots g_n$.

Theorem 16.7. *Let G be a countable group acting non-elementary on a building X of type $\widetilde{A}_2$. Then, $\mathbb{P}(z_n) \to_{n\to\infty} 1$, where z_n is strongly regular hyperbolic isometry.*

Local to global fixed point results:

Theorem 16.8. *Let G be a finitely generated group acting on a building of type $\widetilde{A}_2$. If every element of G fixes a point, then there is a global fixed point.*

Proof. May assume G acts elementarily, otherwise by previous theorem get hyperbolic element which doesn't fix a point (contradiction). Two possibilities: G has a bounded orbit on X, in which case can apply extended Bruhat-Tits fixed point theorem to obtain a fixed point and we are done. Second case: G has bounded orbit on ∂X, which reduces to trees. In this latter case, there exists G_0 which fixes a point of ∂X of finite index in G. Note: if G_0 fixes a point on X, then so does G. We have G_0, g_1G_0, …, g_nG_0, so y, g_1y, …, g_ny yields bounded orbit, so in previous case. If G_0 fixes a point on boundary (asymptotic rays). Strongly asymptotic: bounded distance, eventually coincide. Tree: T_ξ, points are equivalence classes of strongly asymptotic rays.

Question: How to define distance between two points: eventual Euclidean distance can produce this is a tree (difficult), in fact an $\mathbb{R}$-tree. $G \curvearrowright T_\xi$. If $g \in G$, g fixes some $x^g \in X$. Fixes ξ. Fix ray from x^g to ξ. So g fixes $[r]$. All elements of G fix a point in T_ξ, implying G has a global fixed point on T_ξ. □

17. Mapping class groups

Book: Forb-Margalit "A primer of mapping class groups".

Theorem 17.1 (Classification of surfaces finite type)**.** *Every connected orientable surface of finite type is diffeomorphic to some $S_{g,n,b}$, surface obtained by gluing $g \geq 0$ copies tori T^2 with sphere S^2, and removing b open disks and n points. Note Euler characteristic is $\chi(S) = 2 - 2g - n - b$.*

Question: What is Homeo(S), group of self-homeomorphisms?

For example, homeomorphisms of the torus: $T = \mathbb{R}^2/\mathbb{Z}^2$, $A \in \mathrm{SL}(2,\mathbb{Z})$. A determines homeomorphism of torus. There are threee types: periodic (elliptic), reducible (parabolic), Asonov (hyperbolic).

A Dehn twist (cut open cylinder, rotate 2π, glue back), corresponds to

$$\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}\begin{pmatrix}1\\ 0\end{pmatrix} = \begin{pmatrix}1\\ 0\end{pmatrix}$$

and

$$\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}\begin{pmatrix}0\\ 1\end{pmatrix} = \begin{pmatrix}1\\ 1\end{pmatrix}.$$

Theorem 17.2 (Nickson-Thurston, Classification of homeomorphisms of compact surfaces)**.** *Let $F : S \to S$ be an orientation preserving homeomorphism. Then, F is homotopic to either:*

1. *A periodic homeomorphism;*
2. *A reducible homeomorphism;*
3. *Pseudo-Anosov homeomorphism.*

Problem: Homeo(S) is large in the sense it is typically infinite dimensional topological group.

Proposed fixex: quotient out by homotopy, but better by isotopy. $\mathrm{Mod}(S) = \mathrm{Homeo}^+(S, \partial S)/\mathrm{Homeo}_0(S, \partial S)$, orientative-presevering homeomorphisms which pointwise fix the boundary ∂S.

Other possible definitions:

Theorem 17.3 (Munkres, 1950s). *Every homeomorphism of S relative to ∂S is isotopic to a diffeomorphism of S relative to ∂S.*

Context and motivation (Henry Wilton):

(1) $\phi : S \to S \to M_\phi : S \times [0,1]/\sim$, where we identify $(x,1)$ with $(\phi(x),0)$.
(2) Moduli space: Teichmuller space up to action of $\mathrm{Mod}(S)$.
(3) Analogy:

Surfaces	*Tori*
S	T^n
$\pi_1(S)$	$\mathbb{Z}^2$
$\mathrm{Mod}(S)$	$\mathrm{SL}(n,\mathbb{Z})$
Mapping classes	Linear maps
Closed curves	Vectors

The following underpins many computation.

Lemma 17.4 (Alexander Lemma, "Alexander's Trick"). *The mapping class group of the closed disc* $\mathrm{Mod}\big(D^2\big)$*, is trivial.*

Proof. $f : D \to D$ such that $f|_{\partial D} = \mathrm{id}$. Define

$$f_t(x) = \begin{cases} (1-t)f\left(\frac{x}{1-t}\right), & 0 \le |x| \le 1-t; \\ x, & 1-t \le |x| \le 1. \end{cases}$$

□

3-times punctured sphere: can identify with $\mathbb{C} \setminus \{0,1\}$ by removing $0,1,\infty$ from $\widehat{\mathbb{C}}$. Then, $\mathrm{Mod}(S_{0,3}) \cong S_3$.

Have a look at Katie Vokes "An introduction to mapping class group" slides.

Infinite mapping class groups: Consider annulus which can be identified with cylinder $A = S^1 \times [0,1]$. Then, we have the following (note torus is $\mathrm{SL}(2,\mathbb{Z})$).

Proposition 17.5. $\mathrm{Mod}(A) \cong \mathbb{Z}$.

Proof. $\widetilde{A} = [0,1] \times \mathbb{R}$ is the universal cover. Covering map: $\widetilde{A} \to A$, $(x,y) \mapsto (x, \exp(2\pi i y))$. Let $\phi : A \to A$ homeomorphism restricting to identity on ∂A. Then $\widetilde{\phi}$ unique lift of ϕ fixing the origin. Lift of identity on $1 \times S^1$. Translation by integer n. Guaranteed by homotopy lifting property. Set map $\mathrm{Mod}(A) \to \mathbb{Z}$, $[\phi] \mapsto n$. We must show it is a group isomorphism. It is a group HM: $\widehat{\phi \circ \psi} = \widehat{\phi} \circ \widehat{\psi}$. Surjective because $\widetilde{\phi} = \begin{pmatrix} 1 & 0 \\ n & 1 \end{pmatrix}$, related to Dehn twist. Injective: Harder. □

18. Quaternionic Reflection Groups

The quaternions are a 4-dimensional vector space over $\mathbb{R}$ with basis $\{1,i,j,k\}$, represented in the form $a + bi + cj + dk$ $(a,b,c,d \in \mathbb{R})$. Quaternions satisfy the following properties: $i^2 = j^2 = k^2 = ijk = -1$, $ij = k, jk = i, ki = j$, and $ji = -k$, $kj = -i$, $ik = -j$. Since $ij = k$, one has:

$$a + bi + cj + dk = a + bi + cj + dij = (a+bi) + (c+di)j,$$

so the quaternions are also a 2-dimensional vector space over $\mathbb{C}$ with basis $\{1,j\}$.

The set of quaternion basis vectors $\{1,i,j,k\}$ generates a group: Consider the subset of quaternions $\{\pm1, \pm i, \pm j, \pm k\}$. This set is a non-abelian group under the operation of multiplication, and is denoted by Q_8. The group Q_8 can also be represented as a subgroup of $\mathrm{GL}(2,\mathbb{C})$, where

$$1 \mapsto I, i \mapsto \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, j \mapsto \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, k \mapsto \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}.$$

A *complex reflection*, or unitary reflection, is a linear transformation on any finite-dimensional complex vector space which fixes a subspace of dimension $d-1$ of $\mathbb{C}^d$ pointwise with finite order.

A *complex reflection group* is a group that is generated by complex reflections. If G is a group of complex reflections acting on the vector space $\mathbb{F}^d$, then G acts faithfully on $\mathbb{F}^d$.

A *quaternionic reflection* is a linear transformation on a finite-dimensional quaternionic vector space which fixes a subspace of dimension $d-1$ of $\mathbb{H}^d$ pointwise with finite order.

A *quaternionic reflection group* is a group generated by quaternionic reflections. The quaternions are an extension of the complex numbers, so it follows that quaternionic reflections and quaternionic reflection groups are very similar to complex reflections and complex reflection groups, respectively.

All finite quaternionic reflection groups were classified by Cohen in 1980.

A matrix M with quaternion entries $M \in \mathrm{Mat}_n(\mathbb{H})$ can be uniquely written as the suym of two matrices M_1, M_2 with complex entries $M_1, M_2 \in \mathrm{Mat}_n(\mathbb{C})$:

$$M = M_1 + M_2 j.$$

Using this representation, we can map quaternionic matrices to complex matrices as follows:

$$(-)^v : \mathrm{Mat}_n(\mathbb{H}^2) \to \mathrm{Mat}_{2n}(\mathbb{C}), M = M_1 + M_2 j \mapsto \begin{pmatrix} M_1 & -M_2 \\ \overline{M_2} & \overline{M_1} \end{pmatrix}.$$

This mapping is known as *complexification* of quaternionic matrices.

The ring of quaternions H is isomorphic to the ring of matrices with complex entries of the form:

$$A = \begin{pmatrix} x & -y \\ \overline{y} & \overline{x} \end{pmatrix}.$$

This is done by showing

$$a + bi + cj + dk \mapsto \begin{pmatrix} a+bi & -c-di \\ c-di & a-bi \end{pmatrix}$$

is an isomorphism.

Theorem 18.1 (Cohen)**.** *Let G be a finite irreducible proper quaternionic subgroup of $U_n(H)$. Then, G^v is irreducible.*

Proof. Proof outline:

- G acts irreducibly on the quaterionic vector space H. We want to show that this irreducibility is preserved on the complexifed space $H_{\mathbb{C}}$.
- Suppose, for contradiction, that $H_{\mathbb{C}}$ is reducible. Then it can be decomposed into the direct sum of two non-trivial G-invariant subspaces.
- This leads to a contradiction, since it implies that the original quaternionic representation of G on H is reducible.

□

19. Property (T)

Definition 19.1. A *representation* of a group G on a vector space V over $\mathbb{K}$ is a homomorphism $\rho : G \to \mathrm{GL}(V)$.

Definition 19.2. Let $(\mathcal{H}, \langle\cdot\rangle)$ be a Hilbert space. The *unitary group* $\mathcal{U}(\mathcal{H})$ of $\mathcal{H}$ is the group of all invertible linear continuous maps $T : \mathcal{H} \to \mathcal{H}$ which are unitary, that is, preserve the inner product $\langle\cdot\rangle$.

Let G be a group. A *unitary representation* $(\pi, \mathcal{H})$ of G in $\mathcal{H}$ is a group homomorphism $\pi : G \to \mathcal{U}(\mathcal{H})$ that is strongly continuous This means that for every $\xi \in \mathcal{H}$, the map $g \mapsto \pi(g)\xi$ is continuous.

The natural question is whether such unitary representations exist.

Definition 19.3. For a group G, denote by $\ell_2(G)$,

$$\ell_2(G) = \left\{ f : G \to \mathbb{C} \;\middle|\; \sup_{F \subset G, |F| < \infty} \left\{ \sum_{x \in G} |f(x)|^2 \right\} < \infty \right\}.$$

This is a Hilbert space with inner product $\langle f, g \rangle := \sum_{x \in G} f(x)\overline{g(x)}$. If $f \in \ell_2(G)$, then $[f \neq 0]$ is countable so this makes sense.

Definition 19.4. Let G be locally compact. The *left-regular representation* λ_G of G is defined as follows. For each $g \in G$, define $\lambda_G(g) : \ell_2(G) \to \ell_2(G)$ by:

$$(\lambda_G(g)f)(x) = f\big(g^{-1}x\big).$$

One can show this is a unitary representation of G in $\ell_2(G)$.

Definition 19.5. For $i \in I$ an index set, let $\pi_i : G \to \mathcal{U}(\mathcal{H}_i)$ be unitary representations. We can define the direct sum

$$\mathcal{H} = \bigoplus_{i \in I} \mathcal{H}_i = \left\{ (x_i) \;\middle|\; \sum_{i \in I} \|x_i\|^2 < \infty \right\}.$$

With the inner product $\langle (x_i), (y_i) \rangle = \sum_{i \in I} \langle x_i, y_i \rangle$, this is a Hilbert space. There is a natural representation $\pi : \mathcal{U}(\mathcal{H})$ given by:

$$[\pi(g)]((x_i)) = \bigoplus_{i \in I} \pi_i(g)(x_i).$$

Definition 19.6. Let $(\pi, \mathcal{H})$ be a unitrary representation of a locally compact G.

1. Let $\mathcal{Q} \subset G$ and $\epsilon > 0$. A vector $\xi \in B_{\mathcal{H}}$ is *$(\mathcal{Q}, \epsilon)$-invariant* if $\sup_{g \in \mathcal{Q}} \|\pi(g)\xi - \xi\| < \epsilon$.
2. We say π *almost has invariant vectors*, if for every $\mathcal{Q} \subset G$ compact and $\epsilon > 0$, it has $(\mathcal{Q}, \epsilon)$-invariant vectors.
3. $(\pi, \mathcal{H})$ has a *non-trivial fixed invariant vector* if there is an $\xi \neq 0$ such that $\pi(g)\xi = \xi$ for all $g \in G$.

Definition 19.7. Let G be a locally compact topological group. A subset $\mathcal{Q} \subset G$ is a *Kazhdan set*, if there exists $\epsilon > 0$ such that every unitary representation $(\pi, \mathcal{H})$ of G having a $(\mathcal{Q}, \epsilon)$-invariant vector also has a non-zero invariant vector. We say G has *Kazhdan's Property (T)* or is a *Kazhdan group*, if G has a compact Kazhdan set.

Theorem 19.8. *Let G be locally compact. Then, the following are equivalent.*

1. *G has property (T), that is, G has a compact Kazhdan set.*
2. *Every unitary representation of G having almost invariant vectors has invariant vectors.*

Lemma 19.9 (Starting Example)**.** *Let G be a locally compact topological group. Suppose G has a unitary representation $(\pi, \mathcal{H})$ with a $\xi \in B_{\mathcal{H}}$ such that:*

$$\sup_{g \in G} \|\pi(g)\xi - \xi\| < \sqrt{2}.$$

Then, π has invariant vectors.

Corollary 19.10. *Every compact topological group has (T).*

Theorem 19.11 (Compact Generation)**.** *Let G be a locally compact group with property (T). Then, G is compactly generated.*

Corollary 19.12. *A discrete group Γ with (T) is finitely generated.*

Theorem 19.13. *If G has (T), then G/G' is compact.*

Theorem 19.14. *If G has (T) and H is a closed subgroup of G with finite covolume, then H has (T) as well.*

Theorem 19.15. *A lattice Γ in a locally compact group G has Property (T) if, and only if, G has Property (T).*

Example 19.16. Let $\mathbb{K}$ be a local field (e.g., $\mathbb{R}$, $\mathbb{C}$, $\mathbb{Q}_p$). $\mathrm{SL}(n, \mathbb{K})$ for $n \geq 3$ and $\mathrm{Sp}_{2n}(\mathbb{K})$. In general, simple real Lie groups of real rank at least 2 have Property (T).

Non-examples: $\mathrm{SL}(2, \mathbb{R})$ does not have property (T). $\mathbb{R}$ and $\mathbb{Z}$ do not have (T) either.

We now consider some applications. If we consider the converse of (T), if a unitary representation of a Kazhdan group G does not have non-zero invariant vectors.

A notable application of Kazhdan's property (T) is in the area of expander graphs within graph theory. Consider a graph modelling a large communication system between vertices where with the edges representing information propogation in one unit time. Naturally, there are two competing constraints for a real life situation:

(1) We want to minimise the number of edges, minimising the cost of communication channels.
(2) We want every set of vertices to be well connected to the other vertices so we can communicate efficiently.

If we want to model arbitrarily large communication systems, we would like some family of graphs for any vertex number where the number of edges does not grow very fast.

Definition 19.17. Let $c > 0$. A finite k-regular graph $X = (V, E)$ on n vertices is an *(n, k, c)-expander*, if for every subset of vertices $A \subset V$ with $|A| \leq \frac{n}{2}$,

$$\frac{|\partial A|}{|A|} \geq c.$$

Definition 19.18. Let $c > 0$ and $k \in \mathbb{N}$. Let $(X_i) = ((V_i, E_i))$ be a sequence of k-regular graphs such that $|V_i| \to \infty$. We say (X_i) is an *expander family*, if there is a $c > 0$ such that each graph in the sequence is a c-expander.

The existence of such families had been previously proved, but the first explicit construction of such a family was given by Margulis in 1975. His proof employed Kazhdan's Property (T).